String Theory at Finite Coupling

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Based on:

1. A.S., arXiv:1304.0458

2. Christopher Beem, Leonardo Rastelli, A.S., Balt van Rees, arXiv:1306.3228 + work in progress

Seoul, June 2013

Problem with perturbation theory

Generically perturbation expansion in a QFT is an asymptotic expansion.

– has zero radius of convergence.

String perturbation theory is also expected to be an asymptotic expansion.

To gain some insight into asymptotic expansion, consider:

$$F(g) = \int_{-\infty}^{\infty} dx\, e^{-x^2/2 - g^2 x^4/4!} = \frac{\sqrt{3} e^{\frac{3}{4g^2}} K_{\frac{1}{4}}\left(\frac{3}{4g^2}\right)}{g}$$

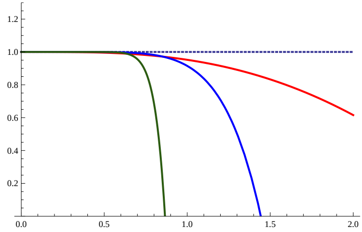
$K_\nu(x)$: Modified Bessel function of the second kind.

Its perturbation expansion in g

$$F(g) = \sum_{n=0}^{\infty} (-1)^n (g^2/4!)^n 2^{2n+\frac{1}{2}} \frac{\Gamma(2n+\frac{1}{2})}{\Gamma(n+1)}$$

has vanishing radius of convergence.

The plot of the ratio of 1st, 3rd and 7th order perturbative results to F(g) as functions of g:



Higher order terms make the result worse at finite coupling.

⇒ asymptotic expansion cannot be directly used for finding even the approximate value of the function at finite coupling.

Since the mid 1990's we have known that many string theories and QFT's have strong-weak coupling duality.

A string theory / QFT with coupling g is equivalent to another (or same) string theory / QFT with coupling 1/g.

Thus the theory for large g can be studied using the dual theory at small g.

Can we use this to address the question of what happens at finite coupling?

Strategy for studying the theory at finite coupling

In a dual pair of theories, any physical quantity F(g) can be studied both at small g and large g.

We have a Taylor series expansion of F(g) at both ends.

Try to combine both informations into one function.

We shall discuss in detail one class of such functions based on fractional power of polynomials (FPP).

But often other functions like multi-point Padé approximants can also be used.

Suppose we know that

$$F(g) = A\, g^a (1 + c_1 g + c_2 g^2 + \cdots + c_m g^m + \cdots) \quad \text{as } g \to 0$$

$$F(g) = B\, g^b (1 + d_1 g^{-1} + d_2 g^{-2} + \cdots d_n g^{-n} + \cdots) \quad \text{as } g \to \infty$$

Then define the approximation $F_{m,n}(g)$ to $F(g)$ to be

$$F_{m,n}(g) = A\, g^a \Big\{ 1 + a_1 g + \cdots a_m g^m + b_n g^{m+1} + \cdots b_1 g^{m+n} + \left(\frac{B}{A}\right)^{\frac{m+n+1}{b-a}} g^{m+n+1} \Big\}^{(b-a)/(m+n+1)}$$

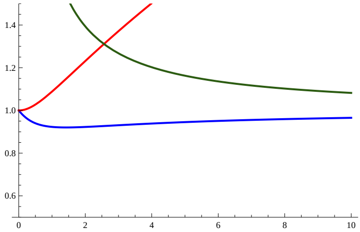
$a_1, \cdots a_m, b_1, \cdots b_n$ are fixed by matching the Taylor series expansion of $F_{m,n}(g)$ at small and large g with those of F(g).

To test the efficiency of this procedure let us go back to

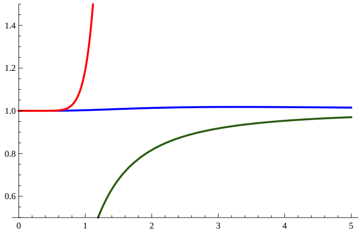
$$F(g) = \int_{-\infty}^{\infty} e^{-x^2/2 - g^2 x^4/4!}\, dx$$

Following the prescription given earlier we can construct the interpolating functions $F_{m,n}(g)$.

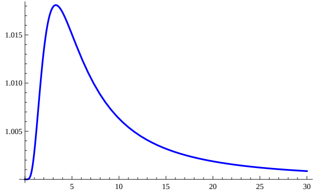
A plot of $F_{0,0}(g)/F(g)$ and also the ratios of 0-th order weak coupling result and 0-th order strong coupling result to F(g).



A plot of $F_{4,1}(g)/F(g)$ and also the ratios of 4-th order weak coupling result and 1-st order strong coupling result to $F(g)$.



A plot of $F_{4,1}(g)/F(g)$ on a different scale.



Lesson: Knowing the function at two ends we have a much better description of the function than knowing it at either end.

This procedure has been tested on many other functions with similar encouraging results

Typically at 5-10% level the difference between successive approximations gives a good estimate of the error.

We shall now apply this insight to string theory and QFT.

SO(32) heterotic string theory contains a stable massive particle transforming in the spinor representation of SO(32).

Its strong coupling dual is the stable non-BPS D0-brane in type I string theory.

Goal: Determine the mass of the stable particle as a function F(g) of the coupling constant g of heterotic string theory.

Known weak coupling expansion (with suitably scaled g and F)

$$\mathbf{F}(\mathbf{g}) = \mathbf{g}^{1/4}(\mathbf{1} + \mathbf{K_w}\,\mathbf{g}^{2} + \mathcal{O}(\mathbf{g}^{4})), \qquad \mathbf{K_w} = .\mathbf{23}$$

Known strong coupling expansion

$$\mathbf{F}(\mathbf{g}) = \mathbf{g}^{3/4}(\mathbf{1} + \mathbf{K_s}\,\mathbf{g}^{-1} + \mathcal{O}(\mathbf{g}^{-2})), \qquad \mathbf{K_s} = .\mathbf{351}$$

Using this we can compute $\mathbf{F_{m,n}}(\mathbf{g})$ for $\mathbf{m} \leq \mathbf{3}$, $\mathbf{n} \leq \mathbf{1}$.

$$F_{0,0}(g) = g^{1/4}\,(1+g)^{1/2}\,,$$

$$F_{0,1}(g) = g^{1/4}\,(1+4\,K_s g + g^2)^{1/4}\,,$$

$$F_{1,1}(g) = g^{1/4}\,(1+6\,K_s g^2 + g^3)^{1/6}\,,$$

$$F_{2,0}(g) = g^{1/4}\,(1+6\,K_w g^2 + g^3)^{1/6}\,,$$

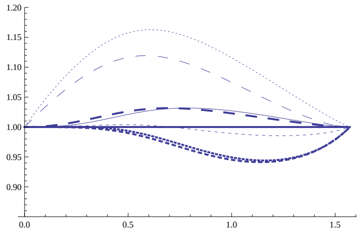
$$F_{2,1}(g) = g^{1/4}\,(1+8\,K_w g^2 + 8K_s g^3 + g^4)^{1/8}\,,$$

$$F_{3,0}(g) = g^{1/4}\,(1+8\,K_w g^2 + g^4)^{1/8}\,,$$

$$F_{3,1}(g) = g^{1/4}\,\left(1+10\,K_w g^2 + 10\,K_s g^4 + g^5\right)^{1/10}\,.$$

In this case we do not have a known result to compare to, but we can get an estimate of the error by comparing results at different orders.

Graph of $F_{m,n}(g)/F_{3,1}(g)$ vs. $\tan^{-1} g$ for different (m,n).



thin dots: $F_{0,0}$, thick dots: $F_{1,0}$, small thin dashes: $F_{2,0}$

small thick dashes: $F_{3,0}$, large thin dashes: $F_{0,1}$,

large thick dashes: $F_{1,1}$, continuous thin line: $F_{2,1}$,

continuous thick line: $F_{3,1}$

Except $F_{0,0}$, all other $F_{m,n}$'s lie within about 10% of $F_{3,1}$.

Other interpolation schemes e.g. Padé approximant also give results within 10% of $F_{3,1}$.

$\Rightarrow$ $F_{3,1}(g)$ most likely gives the mass of the stable non-BPS particle within 10% over the entire range of g.

Application to $\mathcal{N} = 4$ SYM theories with SU(N) gauge group

with Beem, Rastelli, van Rees

Our focus will be on the non-BPS twist two operators:

$$\mathcal{O}_M \equiv \mathrm{Tr}(\phi^I D^M \phi^I), \quad M = 0, 2, 4, \cdots$$

Conformal dimension

$$\Delta_M = 2 + M + \gamma_M$$

Perturbative result for γ_M is known to four loop order for M=0,2 and three loop order for M=4.

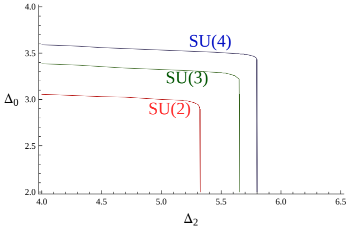
Kotikov, Lipatov, Onishchenko, Velizhanin; Kotikov, Lipatov, Rej, Staudacher, Velizhanin; Fiamberti, Santambrogio, Sieg, Zanon; Bajnok, Janik, Lukowski; Velizhanin

Our goal wil be to compute γ_M at finite N and

$$\tau = \frac{\theta}{2\pi} + i\,\frac{4\pi}{g_{YM}^2} \equiv y + i g^{-1}$$

Beem, Rastelli and van Rees found constraints on Δ_M using conformal bootstrap approach.

The result was a strict upper bound on the anomalous dimensions, e.g. in the $\Delta_0 - \Delta_2$ plane:



The bootstrap approach does not tell us what the dimensions are as function of τ.

Nevertheless BRV suggested that the corner values correspond to the actual values at $\tau = i$ or $\tau = e^{i\pi}/3$

Strategy: Find the strong coupling results using a subgroup of the S-duality symmetry and use them to construct an interpolating function.

We use two subgroups:

Z_2: $\tau \to -1/\tau$ keeps fixed i

Z_3: $\tau \to (\tau - 1)/\tau$ keeps fixed $e^{i\pi/3}$.

Final results inside the fundamental domain of τ are fairly insensitive to which of the two subgroups we use.

Since the S-duality transformation induces a θ-dependent strong coupling expansion, the final interpolating function acquires a θ-dependence.

Results of interpolation:

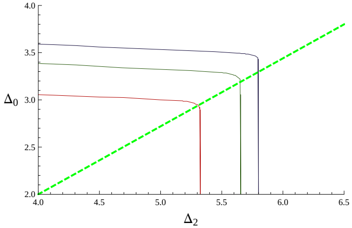
1. The ratios δ_2/δ_0 and δ_4/δ_0 remain almost constant over the whole fundamental domain of τ.

$$\frac{\delta_2}{\delta_0} \simeq \frac{25}{18}, \qquad \frac{\delta_4}{\delta_0} \simeq \frac{49}{30}$$

– given by one loop results.

The estimates for δ_2/δ_0 and δ_4/δ_0 at any τ deviates from these values by less than 1%.

$\Rightarrow$ all the physically allowed Δ_m's lie within a narrow band around a straight line in the space of the Δ_m's.

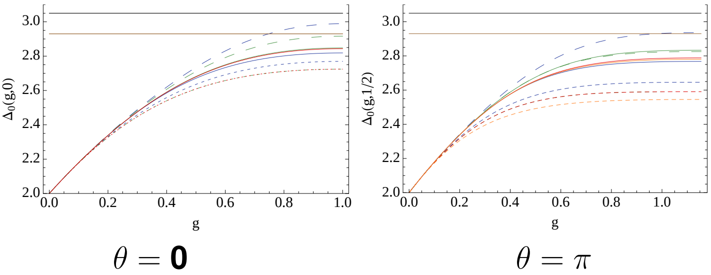


Note: For SU(4) the line seems to miss the corner

– could be either due to inadequacy of our approach at large N or still existing numerical errors in the BRV analysis.

We shall now describe the actual values of Δ_0 for different values of τ.

Result for SU(2)



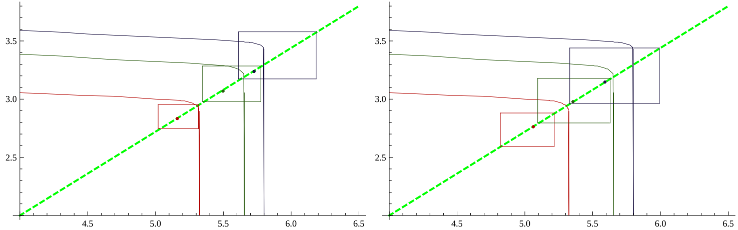
small dash: two loops, large dash: three loops continuous: four loops

different colors: different interpolation schemes (FPP, Pade etc.)

upper line: absolute bound lower line: corner value

– results consistent with BRV bound.

Final summary (with error bars) for estimated values of Δ_m's at $\tau = \mathrm{i}$ and $\tau = e^{i\pi/3}$:



$\tau = \mathrm{i}$ $\qquad$ $\tau = \exp(i\pi/3)$

Note: The errors are only along the green dashed line.

lower/upper error bar: average 2/3 loop result.

BRV conjecture is alive both at $\tau = \mathrm{i}$ and $\tau = e^{i\pi/3}$.

Generalization

1. This method has also been used to compute various physical quantities in the large N limit of SYM theories using AdS/CFT, taking ’t Hooft coupling as the interpolation parameter.

— gives good agreement with exact results where the latter are known.

2. This method has also been used in other physical problems where the behaviour of a function is known at the two ends.

– good agreement with numerical results.

3. In some cases the interpolation method itself requires slight generalization.

e.g. we could take $(F - F_{m,n})$ for some fixed m,n and begin the whole process again.