Complex Chern-Simons from M5 branes on the squashed 3-sphere

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- The squashed 3-sphere partition function of the 3d N=2 theory $\mathcal{T}_{\mathfrak{g}}[M_3]$ given by the (2,0) theory twisted on M_3.

- Supersymmetric backgrounds for 5d maximally supersymmetric Yang-Mills.

- Emergent gauge symmetry from supersymmetry: complex Chern-Simons from reduction on the sphere.

(2,0) theory in six dimensions

- Labeled by ADE "gauge group", describes the dynamics of multiple M5 branes.
- No marginal or relevant deformations, but believed to be a local CFT.
- Strongly coupled, no known Lagrangian description.

(2,0) theory on a circle

- Compactifying on a circle flows in the IR to 5d $\mathcal{N}$=2 YM with gauge group G.

- In 5d, Yang-Mills theory is IR free, and strongly coupled in the UV.

- Instanton-solitons are identified as the KK modes.

- For susy quantities – higher order operators in whatever is the exact 5d theory are plausibly Q-exact.

Recall talks by Douglas, Seok Kim, Vafa

Compactifications

- On Riemann surfaces to 4d $\mathcal{N}$=2 theories
- On 3-manifolds to 3d $\mathcal{N}$=2 theories
- On 4-manifolds to 2d (0,2) theories

Recall talk by Gukov

- A new window on strongly coupled SCFTs in lower dimensions, some which lack Lagrangians.

Curious correspondences

- Observables such as the sphere partition function of the resulting (6−d) dimensional SCFT are equal to the partition function of a particular theory on M_d.

- These 2d Toda and 3d noncompact CS theories are not supersymmetric, don't look like standard gauge theories…

See also Pasquetti's talk

[Alday Gaiotto Tachikawa. Terashima Yamazaki, Dimofte Gaiotto Gukov]

Direct approach

- Find a full 6d background $S \times M$ that preserves supersymmetry.

- Partition function is independent of the size of M due to supersymmetry.

- Reduce on S to find the theory on M.

M5 on $S^3_\ell \times M_3$

$\mathcal{T}_{\mathfrak{g}}[M_3]$ on S^3_ℓ

? on M_3

partition function

Supersymmetry in curved space

- How can one determine the curvature couplings such that susy is preserved?

- Couple to off-shell supergravity, putting the theory in the geometry of interest, and taking M_{Pl} to infinity. Certain background fields in addition to the metric must be turned on to preserve supersymmetry.

Recall Komargodski's talk

[Festuccia Seiberg.
Adams Jockers Kumar Lapan,
Jia Sharpe, ….]

5d intermediary

- Hard to proceed directly from six dimensions, since there is no Lagrangian of the (2,0) theory to reduce. After all, this is why the $\mathcal{T}_{\mathfrak{g}}[M_d]$ theories are interesting.

- Thus one wants to first find a circle isometry to obtain 5d YM in some background – then one can simply derive the theory on M_d.

N M5 branes on a 3-manifold

- Take 3 dimensions to be a small 3-manifold.

- To preserve some supersymmetry, one may take the normal directions to be in the cotangent bundle: $R^{2,1} \times T^*M_3 \times R^2$ is the 11d geometry. $SO(3)_R \times SO(3)$ is broken to the diagonal, resulting in 3d $\mathcal{N}$=2 supersymmetry.

- The IR 3d $\mathcal{N}$=2 CFT is independent of the metric on M_3, and has no flavor symmetries for compact hyperbolic manifolds.

$\mathcal{T}_{\mathfrak{g}}[M_3]$

- Provides a new perspective on 3d SCFTs.
- Abelian CSM Lagrangians have been constructed for some classes of examples, using tetrahedral decomposition.

  [Dimofte Gaiotto Gukov, …]
- Harder for compact 3-manifolds.

3-sphere partition function

- Characterizes the number of degrees of freedom of the CFT.

- Computable from a supersymmetric Lagrangian using localization. Preserves nonconformal $SU(2|1) \times SU(2)$ supersymmetry.

[Kapustin Willett Yaakov, D.J., Hama Hosomichi Lee]

Squashed 3-sphere

- There is a squashing of the 3-sphere, S^3_ℓ , changing the size of the Hopf fiber, and preserving SU(2) × U(1) isometry. By adjusting the R scalar and other 3d background fields, one may preserve SU(2|1) × U(1) supersymmetry.

[Imamura Yokoyama]

- The partition function of an N=2 theory on this space is exactly related to that on the ellipsoid.

$b^2|z_1|^2 + b^{-2}|z_2|^2 = 1$ $\ell = \frac{2}{b+b^{-1}}$ [Hama Hosomichi Lee]

3d-3d conjecture

- Terashima Yamazaki, Dimofte Gaiotto Gukov conjectured that the squashed S^3 partition function of the N M5 on M_3 theory is given by a noncompact CS partition function on M_3 with level determined by the squashing.

- How can this bosonic theory with emergent gauge symmetry arise from reduction of a susy gauge theory?

Complex Chern-Simons theory

$$S = \frac{q}{8\pi}\int \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) + \frac{\tilde{q}}{8\pi}\int \mathrm{Tr}\left(\bar{\mathcal{A}}\wedge d\bar{\mathcal{A}} + \frac{2}{3}\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\right)$$
$$= \frac{k}{4\pi}\int \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A^3 - X\wedge d_A X\right) + \frac{u}{2\pi}\int \mathrm{Tr}\left(\frac{1}{3}X^3 - X\wedge F_A\right)$$

- The Chern-Simons levels are $q = k + iu$ and $\tilde{q} = k - iu$
- Noncompact gauge symmetry $\mathcal{A} \to \mathcal{A} + d_{\mathcal{A}}\, g$, for $g \in \mathfrak{g}_{\mathbb{C}}$

[Witten]

Reality properties

- Can't regulate with YM – wrong sign kinetic term. Subtle to define theory non-perturbatively.

[Witten]

- k is an integer so that e^{iS} is invariant under large gauge transformations.

- u is either real or pure imaginary to obtain a unitary theory. Path integral is oscillatory for real u.

$S^2 \times S^1$

- One may make similar conjectures for the 3d superconformal index of these theories.

[Dimofte Gaiotto Gukov. Dongmin Gang, Eunkyung Koh, Sangmin Lee, Jaemo Park]

- Recently demonstrated by Yagi, and by Sungjay Lee and Yamazaki using similar ideas that one obtains complex Chern-Simons with $k = 0$ and u pure imaginary.

Related works in other configurations:
Witten '11, Fukuda Kawano Matsumiya

M5 on $S^3_\ell \times M_3$

$\mathcal{T}_\mathfrak{g}[M_3]$ on S^3_ℓ

5d YM on $S^2 \times M_3$

? on M_3

partition function

Hopf reduction: 5d Yang-Mills

- One can reduce from S^3 to S^2 along the Hopf fiber.

$$ds^2 = \frac{r^2}{4}\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) + r^2\left(d\psi + \cos^2(\theta/2)d\phi\right)^2$$

- This results in 5d YM on S^2 with graviphoton flux and other background fields.

- Nonrenormalizable theory with $\frac{1}{g_{\mathrm{YM}}^2} = \frac{1}{r}$, but here we actually want small *r*.

Background couplings

- One needs to find the complete 6d background in which supersymmetry is preserved. This involves twisting on the 3-manifold, but something else on the squashed sphere.

- Then reduce to five dimensional background, coupled to the dynamical maximally susy Yang-Mills Lagrangian theory.

5d maximal supergravity

- Only interested here in the gravitino and dilatino supersymmetry variations, whose vanishing gives the conditions for preserving rigid susy.

- Can be obtained by reduction of 6d off-shell conformal sugra [Bergshoeff Sezgin van Proeyen]. Here the R symmetries are both SO(5).

- Generalizes 5d N=1 supergravity of [Kugo Ohashi].

6d →5d, off-shell fields

Field	Interpretation	$sp(4)$	w
$\underline{e}_\mu^a$	Metric	**1**	-1
$\underline{V}_\mu^{mn}$	R Gauge Field	**10**	0
$\underline{T}_{\mu\nu\rho}^{mn}$	Auxiliary 3-form	**5**	-2
$\underline{D}^{mn,rs}$	Auxiliary scalar	**14**	2

Metric reduces to 5d as metric, graviphoton and dilaton. $\underline{e}_\mu^a = \begin{pmatrix} e_\mu^a & e_\mu^5 = \alpha^{-1} C_\mu \\ e_z^a = 0 & e_z^5 = \alpha^{-1} \end{pmatrix}$

$$\underline{V}_a^{mn} \to V_a^{mn}, a \neq 5, \;\; \underline{V}_5^{mn} \equiv S^{mn}, \;\; \underline{T}_{abc}^{mn} \to \underline{T}_{ab5}^{mn} \equiv T_{ab}^{mn}, \;\; \underline{D}^{mn,rs} \to D^{mn,rs}$$

5d Yang-Mills action

$$
\begin{aligned}
S_A &= \frac{1}{8\pi^2}\int \mathrm{Tr}\Big(\alpha F\wedge *F + C\wedge F\wedge F\Big), \\
S_\varphi &= \frac{1}{32\pi^2}\int d^5x\sqrt{|g|}\;\alpha\mathrm{Tr}\Big(\mathcal{D}_a\varphi^{mn}\mathcal{D}^a\varphi_{mn} - 4\varphi^{mn}F_{ab}T^{ab}_{mn} - \varphi^{mn}(M_\varphi)_{mn}^{rs}\varphi_{rs}\Big), \\
S_\rho &= \frac{1}{32\pi^2}\int d^5x\sqrt{|g|}\;\alpha\mathrm{Tr}\Big(\rho_{m\gamma}i\mathcal{D}^\gamma_\beta\rho^{m\beta} + \rho_{m\gamma}(M_\rho)^{mn\gamma}{}_\beta\rho^\beta_n\Big). \\
S_{int} &= \frac{1}{32\pi^2}\int d^5x\sqrt{|g|}\;\alpha\mathrm{Tr}\Big(\rho_{m\alpha}[\varphi^{mn},\rho^\alpha_n] - \frac{1}{4}[\varphi_{mn},\varphi^{nr}][\varphi_{rs},\varphi^{sm}] - \frac{2}{3}S_{mn}\varphi^{mr}[\varphi^{ns},\varphi_{rs}]\Big).
\end{aligned}
$$

- Note the CS term, the F-scalar mixing, and the cubic scalar potential induced by background fields.

Round sphere case

- Note that $H_3 \times S^3$ with equal radii is conformally flat.
- Thus the (2,0) theory can be put canonically on this space.
- $SO(3)_R$ twisting on H_3 leads to $S^3 \times R^3$.

Background sugra on $S^2 \times R^3$

- For general squashing, all background fields are involved. In 5d, there is graviphoton flux on S^2.

$$ds^2 = dx_0^2 + dx_1^2 + dx_2^2 + \left(\frac{r\ell}{2}\right)^2 \left(d\theta^2 + \sin^2(\theta) d\phi^2\right), \; C = \cos^2(\theta/2) d\phi, \; \alpha = 1/r$$

$$T_{\hat{A}BC} = t\varepsilon_{\hat{a}bc}, \; V_{A\hat{B}\hat{C}} = v\varepsilon_{a\hat{b}\hat{c}}, \; S_{\hat{A}\hat{B}} = s\varepsilon_{\hat{x}\hat{y}}, \; D_{\hat{A}\hat{B}} = d\left(\delta_{\hat{a}\hat{b}} - \frac{3}{2}\delta_{\hat{x}\hat{y}}\right)$$

$$t = s = -\frac{\sqrt{1-\ell^2}}{2r\ell^2}, \; v = -\frac{i}{2r\ell^2}, \; d = \frac{3}{2r^2\ell^2}\left(1 + \frac{1}{\ell^2}\right)$$

Reduction on S^2

- In the limit of a small sphere, the light fields that survive the dimensional reduction are S^2 constant modes of the gauge field (along the 3-manifold) and the scalars, and particular modes of the fermions that transform in the 2 of the SU(2) rotations of the sphere.

- The fermionic action is not diagonalized by the mass basis, so one must include some massive modes and integrate them out.

Gauge sector

- In 5d, the graviphoton induces a Chern-Simons term.

$$S_A = \frac{1}{8\pi^2} \int_{\mathbb{R}^3 \times S^2} \left(\alpha \ \mathrm{Tr}(F \wedge *F) + G \wedge CS(A)\right)$$

- S^2 is simply connected, one just reduces to 3d.

$$S_A = \frac{r\ell^2}{8\pi} \int_{\mathbb{R}^3} \mathrm{Tr}(F \wedge *F) + \frac{1}{4\pi} \int_{\mathbb{R}^3} CS(A)$$

- The 3d YM term disappears in the r $\longrightarrow$ 0 limit.

Scalar action

- The 5 scalars decompose into (3,1) + (1,2) under R-symmetry breaking SO(5) to SO(3) × SO(2).

- The 5d sugra induced masses and those from the R gauge field covariant derivative cancel.

$$S_X = \frac{r\ell^2}{8\pi}\int d^3x \mathrm{Tr}\left(\nabla_a X_b \nabla_a X_b\right) + \frac{1}{4\pi}\int d^3x\, i\varepsilon_{abc}\mathrm{Tr}\left(X_a \nabla_b X_c - i\sqrt{1-\ell^2}X_a F_{bc}\right)$$

$$S_Y = \frac{r\ell^2}{8\pi}\int d^3x \mathrm{Tr}\left(\nabla_a Y_z \nabla_a Y_z\right)$$

Fermion action

- Expanding the twisted fermions in terms of scalars and 1-forms in M_3, a doublet of modes on S^2, and keeping track of the $SO(2)_R$ index:

$$\rho^m = \varepsilon^{\alpha\hat{\alpha}}\lambda^{\sigma\hat{\sigma}} + (\gamma^a)^{\alpha\hat{\alpha}}\,\xi_a^{\sigma\hat{\sigma}}$$

$$S_{ferm} = \frac{1}{32\pi^2}\int d^3x\ \mathrm{Tr}\left[\left(\ \xi_a^{i\hat{i}}\varepsilon_{ij}B_{\hat{i}\hat{j}} - e\widetilde{\xi}_a^{i\hat{i}}B_{ij}\varepsilon_{\hat{i}\hat{j}}\right) i\nabla_a\lambda^{j\hat{j}}\right.$$
$$\left. - \frac{i}{r\ell}\left(\ \xi_a^{i\hat{i}}\xi_a^{j\hat{j}} - \widetilde{\xi}_a^{i\hat{i}}\widetilde{\xi}_a^{j\hat{j}}\right)\varepsilon_{ij}B_{\hat{i}\hat{j}} - \frac{4i}{r\ell^2(1+\ell)}\widetilde{\lambda}^{i\hat{i}}\widetilde{\lambda}^{j\hat{j}}\varepsilon_{ij}B_{\hat{i}\hat{j}}\right]$$

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad e = \sqrt{\tfrac{1-\ell}{1+\ell}}$$

Non-abelian interactions

- These come from the standard quartic terms and Yukawa couplings, and the background induced cubic scalar potential.

$$S_{pot} = \frac{r\ell^2}{8\pi}\int d^3x\ \mathrm{Tr}\left(\frac{1}{2}[X_a, X_b][X_a, X_b] + [X_a, Y_z][X_a, Y_z] + \frac{1}{2}[Y_z, Y_w][Y_z, Y_w]\right)$$

$$+\frac{i\sqrt{1-\ell^2}}{12\pi}\int d^3x\ i\varepsilon_{abc}\mathrm{Tr}\left(X_a[X_b, X_c]\right)$$

$$S_{yuk} = \frac{1}{32\pi^2}\int d^3x\ \mathrm{Tr}\left(\widetilde{\xi}_a^{i\hat{i}}[X_a, \lambda^{j\hat{j}}]B_{ij}\varepsilon_{\hat{i}\hat{j}} - e\xi_a^{i\hat{i}}[X_a, \lambda^{j\hat{j}}]\varepsilon_{ij}B_{\hat{i}\hat{j}} + \left(\frac{2}{1+\ell}\right)\widetilde{\lambda}^{i\hat{i}}[Y_z, \lambda^{j\hat{j}}]B_{ij}\kappa^z_{\hat{i}\hat{j}}\right)$$

Complex connection

- By changing the contour of integration of X, it is natural to define a complex 1-form,

$\mathcal{A}$ = A + i X.

- The action looks almost invariant under a new symmetry, $A \to A - [X, g]$, $X \to X + dg + [A, g]$

except for the term $(D_\mu^A X^\mu)^2$

Obtaining ghosts

- Integrating out the fermions whose mass diverges in the small sphere limit leads to a second order action for the 4 massless fermions.

$$S_\lambda = \frac{ir\ell^2}{64\pi^2(1+\ell)} \int d^3x \; \mathrm{Tr} \Big(\nabla_a \lambda^{i\hat{i}} \nabla_a \lambda^{j\hat{j}} \varepsilon_{ij} B_{\hat{i}\hat{j}} + [X_a, \lambda^{i\hat{i}}][X_a, \lambda^{j\hat{j}}] \varepsilon_{ij} B_{\hat{i}\hat{j}} - \frac{1}{2}[Y_z, \lambda^{i\hat{i}}][Y_w, \lambda^{j\hat{j}}] \Big(\delta^{zw} \varepsilon_{ij} B_{\hat{i}\hat{j}} + i\varepsilon^{zw} \varepsilon_{ij} \varepsilon_{\hat{i}\hat{j}} \Big) \Big)$$

- Non-linear ghost action is Q-equivalent to a quadratic action.

Faddeev-Popov

- The Faddeex-Popov determinant for fixing the noncompact part of the gauge symmetry with the gauge fixing term $(D^A_\mu X^\mu)^2$ is precisely the fermionic determinant!

$$\delta\left(D_\mu X^\mu\right) = D_A^2 g + \left(\mathrm{ad}_X\right)^2 g$$

- There are 4 rather than 2 fermions, and the doubling is exactly cancelled by the 1-loop determinant of the Y scalars.

Complex Chern-Simons

- Therefore,

$$Z_{S^3_\ell}[T_{\mathfrak{g}}(M_3)] = Z_{M_3}[CS_{\mathfrak{g}_C}(1, \sqrt{1-\ell^2})]$$

- Note that both branches of unitary reality conditions for the level u appear.