The M5-brane superconformal index

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Hee-Cheol Kim, S.K. 1206.6339; Hee-Cheol Kim, Joonho Kim, S.K. 1211.0144;
Hee-Cheol Kim, Kimyeong Lee 1210.0853;
Hee-Cheol Kim, S.K., Sung-Soo Kim, Kimyeong Lee to appear.

some related works:
Kallen, Zabzine 1202.1956; Hosomichi, Seong, Terashima 1203.0371;
Kallen, Qiu, Zabzine 1206.6008; Kallen, Minahan, Nedeline, Zabzine 1207.3763;
Imamura 1209.0561, 1210.6308; Lockhart, Vafa 1210.5909;
Hee-Cheol Kim, S.K., Eunkyung Koh, Kimyeong Lee, Sungjay Lee 1110.2175;
Haghighat, Iqbal, Kozcaz, Lockhart, Vafa 1305.6322.

Motivation

- 6d (2,0) SCFT is very unique: "tensor" theory, N^3 degrees, QFT in high dimension, …
- Without a microscopic formulation, we still want to learn something useful about it from effective descriptions + some power of SUSY.

- Reduce on S^1, **5d SYM** at low E: UV degrees from instantons (D0 on D4 = KK modes)

$$F_{\mu\nu} = \star_4 F_{\mu\nu} \quad \text{on } R^4 \qquad \frac{4\pi^2}{g_{YM}^2} = \frac{1}{r_1}$$

- BPS observables: SUSY path integrals are well-defined (~ Gaussian path integral)
- To probe interesting UV physics, one should carefully choose the BPS quantity.

- Index on $S^5 \times S^1$ from SYM on S^5 , $CP^2 \times S^1$, obtained by naïve S^1 reductions: can compute 6d spectrum, also highlighting some limitations/ambiguities of this approach.

(A more "conservative" or "modest" attitude than [Douglas] [Lambert, Papageorgakis, Schmidt-Sommerfeld]. But our calculations may be viewed as a special case of their proposal.)

The index for 6d (2,0) theory

- Put the theory on $S^5 \times R$: energy E ; SO(6) j_1, j_2, j_3 ; $SO(5)_R$ R_1, R_2
- Choose a pair of Q, S (= Q^+) among 32

$$Q^{(R_1,R_2)}_{(j_1,j_2,j_3)} \to Q^{(\frac{1}{2},\frac{1}{2})}_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} \;:\; \text{BPS bound } E = 2R_1 + 2R_2 + j_1 + j_2 + j_3$$

[Kinney, Maldacena, Minwalla, Raju] [Bhattacharya, Bhattacharyya, Minwalla, Raju]

- Index partition function on $S^5 \times S^1$: counts local BPS operators on R^6

$$I(\beta, m, \gamma_1, \gamma_2) = \text{Tr}\left[(-1)^F e^{-\beta'\{Q,S\}} e^{-\beta(E-\frac{R_1+R_2}{2})} e^{\beta m(R_1-R_2)} e^{-\gamma_1(j_1-j_3)} e^{-\gamma_2(j_2-j_3)}\right]$$

- $CP^2 \times S^1$ QFT: 6d fugacities are 5d fugacities (time S^1 is explicit in 5d)
- S^5 interpretation: naïve reduction on S^1 with twistings

1. β ~ S^1 radius ~ 5d or "type IIA" coupling: $\frac{4\pi^2}{g_{YM}^2} = \frac{1}{r_1} = \frac{2\pi}{r\beta}$
2. m: hypermultiplet mass (Scherk-Schwarz reduction) $\frac{\partial}{\partial\tau} \to \frac{\partial}{\partial\tau} - ia_i\frac{\partial}{\partial\phi_i} + \frac{R_1+R_2}{2} - m(R_1-R_2)$
3. a_i = (a, b, c), satisfying a+b+c=0, squash S^5: $e^{-\gamma_1(j_1-j_3)}e^{-\gamma_2(j_2-j_3)} = e^{-\beta(aj_1+bj_2+cj_3)}$

SYM on S^5

- Fields of maximal SYM: w/ SU(1|1), SU(4|1) (at $a_i = 0$), SU(4|2) (at $a_i = 0$, m = ½ or - ½)

- Data needed to construct SYM on S^5: $n_1^2+n_2^2+n_3^2 = 1$

background: $$ds_5^2 = dn_i^2 + n_i^2 d\phi_i^2 + \alpha^2 (a_i n_i^2 d\phi_i)^2 \qquad \alpha^{-2} = 1 - a_i^2 n_i^2 \qquad C = i\sum_{i=1}^{3} a_i n_i^2 d\phi_i$$

Killing spinor eqn: $$\left[\nabla_\mu - C_\mu \sigma_3 - \frac{i}{8\alpha}(dC)^{\nu\rho}\gamma_{\mu\nu\rho}\right]\epsilon = i\gamma_\mu \left[\alpha\sigma_3 - \frac{1}{4\alpha}(dC)_{\nu\rho}\gamma^{\nu\rho} + \frac{i}{2\alpha}\nabla_\nu \alpha \gamma^\nu\right]\epsilon$$

- Hybrid of SUGRA [Festuccia,Seiberg] + brutal [Hosomichi,Seong,Terashima] methods

- Action with off-shell SU(1|1): (bosonic) $\quad D = 2(a_1^2+a_2^2+a_3^2)\alpha^2 \qquad V_{ab} = (dC)_{ab}$

$$\begin{aligned} g_{YM}^2 e^{-1}\mathcal{L} \;=\; & \frac{1}{2}\left(\frac{3}{16\alpha^2}V^2 + \frac{1}{4}R + D\right)\alpha\phi^2 - \frac{1}{4\alpha}\phi^2 V^2 - \frac{1}{2}\phi V^{ab}F_{ab} \\ & -2\phi\left(-\frac{1}{4}V^{ab}F_{ab} - \frac{1}{2}\partial^a \alpha D_a \phi + \frac{i}{4}\alpha^2(\sigma^3)_{ij}D^{ij}\right) \\ & -\alpha\left(-\frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}D^a\phi D_a\phi - \frac{1}{4}D_{ij}D^{ij}\right) + e^{-1}\frac{i}{8}\epsilon^{\mu\nu\lambda\rho\sigma}C_\mu F_{\nu\lambda}F_{\rho\sigma} \end{aligned}$$ } vector multiplet

$$+|D_\mu q^i|^2 + \left(4 - \frac{\alpha^2}{4}\right)|q^i|^2 - \bar{F}_{i'}F^{i'} + ([\bar{q}_i,\phi] - im\alpha\bar{q}_i)([\phi,q^i] - im\alpha q^i) - \bar{q}_i(\sigma^I)^i{}_j([D^I,q^j] + m\alpha^2\delta_3^I q^j)$$ } adjoint hypermultiplet

- A 5d ambiguity: constant shift at inverse-powers of g_{YM}^2 to be added & tuned.

$$\Delta S_0 = \frac{1}{g_{YM}^2}\left[a_1\int R^2 + \text{other background fields}\right] + \frac{1}{g_{YM}^6}\left[a_3\int R + \text{other background fields}\right]$$

~ T $\qquad\qquad$ ~ T^3 (maximally possible high T growth)

to account for high T asymptotics of "free energy"

Localization of path integral

- Localization: $Z(\beta) = \int e^{-S-tQV}$: t independent (V chosen s.t. $\{Q^2,V\}=0$)

- Saddle points: firstly, on round S^5

$$\xi^\mu = \epsilon^\dagger \gamma^\mu \epsilon = \sum_{i=1}^{3} \partial^\mu_{\phi_i}$$

self-dual instantons on CP^2 base

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta\gamma}F^{\alpha\beta}\xi^\gamma \,, \quad F_{\mu\nu}\xi^\nu = 0 \,, \quad D_\mu\phi = 0 \,, \quad D = i\phi\sigma^3$$

all other fields = 0

- Squashed S^5: instantons collapsed to "3 fixed points" (of squashed CP^2).

$$(n_1, n_2, n_3) = (1,0,0), \ (0,1,0) \ \text{or} \ (0,0,1)$$

$U(1)^2$ fixed points on (squashed) CP^2

- solution away from fixed points: a matrix for scalar expectation value

$$\tilde{\xi}^\mu = \sum_{i=1}^{3} a_i \partial^\mu_{\phi_i}$$

$$\phi = \alpha\phi_0 \,, \quad D = i\alpha^2\phi_0\sigma_3 \,, \quad F_{\mu\nu} = \phi_0 V_{\mu\nu} = i\phi_0(d\tilde{\xi})_{\mu\nu}$$

- determinant around saddle point: use suitable index theorems

Result

- Result: determinant factorized to contributions from 3 fixed points $\lambda = r\phi$

$$Z(\beta, m, a_i) = \frac{1}{|W|}\int_{-\infty}^{\infty}\left[\prod_{i=1}^{r} d\lambda_i\right]\exp\left[-\frac{2\pi^2 \mathrm{tr}\lambda^2}{\beta(1+a)(1+b)(1+c)}\right] Z^{(1)}_{\mathrm{pert}} Z^{(1)}_{\mathrm{inst}} \cdot Z^{(2)}_{\mathrm{pert}} Z^{(2)}_{\mathrm{inst}} \cdot Z^{(3)}_{\mathrm{pert}} Z^{(3)}_{\mathrm{inst}}$$

(W: Weyl group, r: rank)

- Z_{pert} 's combine to: see also [Lockhart, Vafa] [Imamura]

$$\det{}_{\mathrm{V}} = \prod_{\alpha\in\mathrm{root}}\prod_{p,q,r=0}^{\infty}\Big(p(1+a)+q(1+b)+r(1+c)+\alpha(\lambda)\Big)\Big((p+1)(1+a)+(q+1)(1+b)+(r+1)(1+c)+\alpha(\lambda)\Big)$$

vector multiplet

$$\det{}_{\mathrm{H}} = \prod_{\mu\in\mathrm{weight}}\prod_{p,q,r=0}^{\infty}\left(p(1+a)+q(1+b)+r(1+c)+m+\frac{3}{2}+\mu(\lambda)\right)^{-1}$$
$$\times\left((p+1)(1+a)+(q+1)(1+b)+(r+1)(1+c)-m-\frac{3}{2}+\mu(\lambda)\right)^{-1}$$

hyper in real representation (e.g. adjoint)

- $Z^{(i)}_{\mathrm{inst}} \sim Z_{\mathrm{Nekrasov}}$ on Omega-deformed $R^4 \times S^1$ (with parameter match): For U(N),

$$Z^{(3)}_{\mathrm{inst}} = \sum_{k=0}^{\infty} e^{-\frac{4\pi^2 k}{\beta(1+c)}}\frac{(1+c)^{-k}}{k!}\oint\left[\prod_{I=1}^{k}\frac{d\phi_I}{2\pi}\right]\prod_{I=1}^{k}\prod_{i=1}^{N}\frac{\sin\pi\frac{\phi_I-\lambda_i-m-\frac{3(1+c)}{2}}{1+c}\sin\pi\frac{\phi_I-\lambda_i+m+\frac{3(1+c)}{2}}{1+c}}{\sin\pi\frac{\phi_I-\lambda_i-\epsilon_+}{1+c}\sin\pi\frac{\phi_I-\lambda_i+\epsilon_+}{1+c}}$$
$$\times\prod_{I\neq J}\sin\pi\frac{\phi_{IJ}}{1+c}\prod_{I,J}\frac{\sin\pi\frac{\phi_{IJ}+3c}{1+c}}{\sin\pi\frac{\phi_{IJ}-a+c}{1+c}\sin\pi\frac{\phi_{IJ}-b+c}{1+c}}\cdot\frac{\sin\pi\frac{\phi_{IJ}+m-\frac{1}{2}+a}{1+c}\sin\pi\frac{\phi_{IJ}+m-\frac{1}{2}+b}{1+c}}{\sin\pi\frac{\phi_{IJ}+m+\frac{3}{2}}{1+c}\sin\pi\frac{\phi_{IJ}+m+\frac{3}{2}+3c}{1+c}}$$

- Weak-coupling expansion at β<<1. Re-expansion with $e^{-\beta}$ at strong coupling?

An unrefined 6d index (w/ 16 SUSY)

- Path integral w/ enhanced SUSY at $a_i = 0$, $m = ½$ or $- ½$ (maximal SYM).
- Vector/hyper cancelation. Final expression can be easily re-expanded.
- Full partition function for U(N), SO(2N): tune constant shift $e^{-\Delta S_0} = e^{\frac{\pi^2 N}{6\beta}}$ to make it an index

  (c_2: dual Coxeter number, |G|: dimension)

$$Z^{U(N)} = e^{\beta\left(\frac{N(N^2-1)}{6}+\frac{N}{24}\right)} \boxed{\prod_{n=0}^{\infty}\prod_{s=1}^{N}\frac{1}{1-e^{-\beta(n+s)}}}$$

$$Z^{SO(2N)} = e^{\beta\left(\frac{c_2|G|}{6}+\frac{N}{24}\right)} \boxed{\prod_{n=0}^{\infty}\left[\frac{1}{1-e^{-\beta(n+N)}}\prod_{s=1}^{N-1}\frac{1}{1-e^{-\beta(n+2s)}}\right]}$$

- $Z[S^5]$ is indeed an index: evidence of the emergence of the "time circle"
- Counts ½ BPS operators w/ one kind of holomorphic derivatives
- Analogous sector: 4d N=4 SYM [Mandal, Suryanarayana] [Grant, Grassi, S.K., Minwalla] (a small part of X. Yin's)

  U(N): $(\partial_1)^n \mathrm{tr} Z^s$ and their multiplications

  SO(2N): $(\partial_1)^n \mathrm{tr} Z^{2s}$, $(\partial_1)^n \sqrt{\det Z}$ and their multiplications

- So not surprisingly, large N limits agree w/ SUGRA on $AdS_7 \times S^4$ & $AdS_7 \times S^4/Z_2$.

The vacuum Casimir "energies"

- The overall prefactor: $e^{-\beta\epsilon_0} \equiv e^{\beta\left(\frac{c_2|G|}{6}+\frac{r}{24}\right)}$
- Casimir "energy": index version (depends on regulator/renormalization)

$$\epsilon_0 = \lim_{\beta'\to 0} \mathrm{tr}\left[(-1)^F \frac{E}{2} e^{-\beta' E}\right] \qquad (\epsilon_0)_{\text{index}} = \lim_{\beta'\to 0} \mathrm{tr}\left[(-1)^F \frac{E-R_1}{2} e^{-\beta'(E-R_1)}\right]$$

- The 6d calculation for the Abelian theory concretely illustrates the difference.

$$\sum_{n=1}^{\infty} \frac{n}{2} \qquad (\epsilon_0)_{\text{index}} = -\frac{1}{24} \neq -\frac{25}{384} = \epsilon_0$$

- Also, G = U(N) at large N: but both show N^3 scalings

$$(\epsilon_0)_{\text{index}} = -\frac{N^3}{6} \neq (\epsilon_0)_{\text{gravity}} = -\frac{5N^3}{24}$$ from AdS_7 dual [Awad, Johnson]

- AdS_7 dual of the index version? (SUSY should constrain all the steps of calculation)

- **Note**: The above Casimir "energy" is uniquely constrained by maximal SU(4|2) SUSY.
- With all 4 chemical potentials turned on, SU(1|1) seems too small to do the same job.
- So generally, we might have implicitly fixed a small ambiguity in the SUSY path integral.
- Any such ambiguities are expected to disappear at maximal SUSY points.

SUSY QFT on CP^2 x R

- SUSY reductions along S^1/Z_K Hopf fiber of S^5/Z_K [H.-C.Kim, K. Lee]

  2π/K rotation with $k \equiv j_1 + j_2 + j_3 + \frac{3}{2}(R_1 + R_2) + n(R_1 - R_2)$

- The 5d theory is a low energy effective QFT for large K.
- Half-an-odd integer n: different twisted reductions, infinitely many 5d QFT
- Our main interest: strong-coupling QFT at K=1, instantons provide KK towers

- On-shell (Euclidean) action: can make SU(1|1) off-shell (only consider U(N))

$$
\begin{aligned}
S &= \frac{1}{g_{YM}^2}\int d^5x\sqrt{g}\,\mathrm{tr}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi^I D^\mu\phi^I - \frac{i}{2}\lambda^\dagger\gamma^\mu D_\mu\lambda - \frac{1}{4}[\phi^I,\phi^J]^2 - \frac{i}{2}\lambda^\dagger\hat\gamma^I[\lambda,\phi^I]\right. \\
&\quad + \frac{2}{r^2}(\phi_I)^2 - \frac{1}{2r^2}(M_n\phi^I)^2 + \frac{1}{8r}\lambda^\dagger J_{\mu\nu}\gamma^{\mu\nu}\lambda - \frac{i}{2r}\lambda^\dagger M_n\lambda - \frac{i}{r}(3-2n)[\phi^1,\phi^2]\phi^3 - \frac{i}{r}(3+2n)[\phi^4,\phi^5]\phi^3 \\
&\quad \left. - \frac{i}{2r}\epsilon^{\mu\nu\lambda\rho\sigma}\left(A_\mu\partial_\nu A_\lambda - \frac{2i}{3}A_\mu A_\nu A_\lambda\right)J_{\rho\sigma}\right]
\end{aligned}
$$

$$M_n \equiv \tfrac{3}{2}(R_1+R_2)+n(R_1-R_2)$$

- SUSY: SU(3|1) x SU(1|1) for n= ½ , -½ ; SU(1|2) for n = 3/2, -3/2; SU(1|1) otherwise

Index on $CP^2 \times S^1$

- It is manifestly an index.
- Saddle point structure is more complicated than S^5.

$$D^1 = D^2 = 0\ , \quad \boxed{F^- = 2sJ}\ , \quad \phi + D = 4s$$

uniform anti-self-dual instantons allowed on CP^2, proportional to Kahler 2-form

- After a localization calculus, one obtains a contour integral: U(N)

$$\boxed{\sum_{s_1, s_2, \cdots s_N = -\infty}^{\infty}} \frac{1}{|W_s|} \boxed{\oint \left[\frac{d\lambda_i}{2\pi}\right]} e^{\frac{\beta}{2}\sum_{i=1}^N s_i^2 - i\sum_i s_i \lambda_i} Z_{\rm pert}^{(1)} Z_{\rm inst}^{(1)} \cdot Z_{\rm pert}^{(2)} Z_{\rm inst}^{(2)} \cdot Z_{\rm pert}^{(3)} Z_{\rm inst}^{(3)}$$

sum over anti-self-dual fluxes

integral over S^1 holonomies (the contour is nontrivial, however)

- $Z_{\rm pert}$:

$$Z_{\rm pert}^{(1)} Z_{\rm pert}^{(2)} Z_{\rm pert}^{(3)} = \prod_{\alpha \in \Delta_+} \frac{\prod_{\sum_{i=1}^3 p_i = \alpha(s)} 2 \sin \frac{\alpha(\lambda + i\sigma) + \beta p_i a_i}{2} \cdot \prod_{\sum_{i=1}^3 p_i = \alpha(s) - 3} 2 \sin \frac{\alpha(\lambda + i\sigma) + \beta p_i a_i}{2}}{\prod_{\sum_{i=1}^3 p_i = \alpha(s) - 1} 2 \sin \frac{\alpha(\lambda + i\sigma) + \beta p_i a_i - \beta \hat{m}}{2} \cdot \prod_{\sum_{i=1}^3 p_i = \alpha(s) - 2} 2 \sin \frac{\alpha(\lambda + i\sigma) + \beta p_i a_i + \beta \hat{m}}{2}}$$

- $Z_{\rm inst}$: product of 3 Nekrasov's $Z_{\rm inst}$ on $R^4 \times S^1$, with suitable identifications of parameters

- This index has subtle structures: vacuum comes with nonzero anti-self-dual flux.

$$s = (s_1, s_2, \cdots, s_N) = (N-1, N-3, N-5, \cdots, -(N-1))$$

contributes to vacuum "energy" by $\quad \epsilon_0 \leftarrow -\frac{N(N^2-1)}{6}$

Some tests

(We work with 5d QFT at n= -½ , enjoying some nicer properties) (instanton number ~ energy)

- U(N) index agrees w/ large N gravity dual for k $\leq$ N: checked for N $\leq$ 3
- E.g. k = N = 3: (all multiplied by vacuum energy factor & q^3) $q = e^{-\beta},\ y_i = e^{-\beta a_i},\ y = e^{\beta(m-\frac{1}{2})}$

$$\begin{aligned}
Z_{(2,0,-2)} &= 3\left[y^2(y_1+y_2+y_3)+y(y_1^2+y_2^2+y_3^2)+y^{-1}(y_1+y_2+y_3)-(1+\frac{y_1}{y_2}+\frac{y_2}{y_1}+\cdots)+y^3\right] \\
&\quad +6y\left[y(y_1+y_2+y_3)-(y_1^{-1}+y_2^{-1}+y_3^{-1})+y^{-1}+y^2\right]+y^3 \\
Z_{(2,-1,-1)}+Z_{(1,1,-2)} &= -2y\left[y(y_1+y_2+y_3)-(y_1^{-1}+y_2^{-1}+y_3^{-1})+y^{-1}+y^2\right] \\
&\quad -2y\left[y(y_1+y_2+y_3)-(y_1^{-1}+y_2^{-1}+y_3^{-1})+y^{-1}+y^2\right] \\
&\quad -4y^3-4y^2(y_1+y_2+y_3)-2y\left(y_1^2+y_2^2+y_3^2-\frac{1}{y_1}-\frac{1}{y_2}-\frac{1}{y_3}\right)+2\left(\frac{y_1}{y_2}+\frac{y_2}{y_3}+\cdots\right)-2y^{-1}(y_1+y_2+y_3) \\
Z_{(1,0,-1)} &= y^3+y^2(y_1+y_2+y_3)-y(y_1^{-1}+y_2^{-1}+y_3^{-1})+1
\end{aligned}$$

add all

$$Z_{SUGRA} = 3y^3+2y^2(y_1+y_2+y_3)+y\left(y_1^2+y_2^2+y_3^2-\frac{1}{y_1}-\frac{1}{y_2}-\frac{1}{y_3}\right)-\left(\frac{y_1}{y_2}+\frac{y_2}{y_1}+\cdots\right)+y^{-1}(y_1+y_2+y_3)$$

- General U(N) index up to k$\leq$2. Large N agrees w/ SUGRA: e.g. k=2 example

Contributions from various anti-self-dual fluxes

$$\left\{\begin{aligned}
&q^2\left[\frac{N(N+1)}{2}y^2+Ny(y_1+y_2+y_3)-N\left(y_1^{-1}+y_2^{-1}+y_3^{-1}\right)+Ny^{-1}\right] \\
&-(N-1)(N-2)q^2y^2-(N-1)q^2\left[y^2+y(y_1+y_2+y_3)-(y_1^{-1}+y_2^{-1}+y_3^{-1})+y^{-1}\right] \\
&+\frac{(N-2)(N-3)}{2}q^2y^2
\end{aligned}\right.$$

$$= q^2\left[2y^2+y(y_1+y_2+y_3)-(y_1^{-1}+y_2^{-1}+y_3^{-1})+y^{-1}\right]$$

SUGRA index on $AdS_7 \times S^4$

- Of course new predictions of spectrum at k>N beyond SUGRA.

Concluding remarks

All supporting evidence we found

maximal SUSY point w/ 1 fugacity

systematically justified w/ 4 fugacities: low E expansion

"$Z[S^5]$ = $\text{Index}[S^5 \times S^1]$ = $\text{Index}[CP^2 \times S^1]$"

Also justified from Abelian index w/ 4 fugacities [Lockhart, Vafa] [H.-C. Kim, J. Kim, S.K.]

- $CP^2 \times R$ QFT approach could be useful for studying **6d (1,0) SCFT's**.
- Other applications of $Z[S^5]$ (w/ different matter contents): 5d SCFT [Jafferis, Pufu] [Assel, Estes, Yamazaki]; Relation to 2d CFT's correlators [Nieri, Pasquetti, Passerini]
- A more concrete formulation of **any** nontrivial higher dimensional QFT (e.g. SCFT's in d=5,6) would be desirable, many of them predicted by string theory.