Supersymmetry beyond Flat Space

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To Appear

Introduction

Supersymmetry in flat space leads to detailed understanding of the dynamics of many strongly coupled gauge theories.

Some powerful methods for analyzing such theories have been

- Chiral operators, chiral ring.
- General aspects of QFT, such as anomalies.

Introduction

In recent years it has been realized that there is a lot more to be uncovered. Roughly speaking, operators that are annihilated by one supercharge can be counted in $\mathcal{N}=1$ supersymmetric theories on $\mathbb{S}^3\times\mathbb{S}^1$. This leads to powerful checks of dualities and other dynamical properties of QFT.

Three-dimensional $\mathcal{N}=2$ theories on $\mathbb{S}^3$ have also led to a proliferation of new exact results and insights regarding renormalization group flows, entanglement entropy, etc.

[Romelsberger, Pestun, Kapustin-Willet-Yaakov, Jafferis,...]

Introduction

- Powerful constraints on dynamics, extending the tools beyond those readily available in flat space.
- It would be nice to understand what the partition function $Z_{\mathcal{M}_d}$ computes and what it depends on – "what is the four-supercharges analog of Donaldson theory and topological twisting?"
- Many curious properties of the partition function were observed in the cases $\mathbb{S}^3$ and $\mathbb{S}^3 \times \mathbb{S}^1$ (holomorphy, decomposition into blocks, $SL(3,\mathbb{Z})$....) and it is likely that to understand them we need to embed these special cases in a general picture.

  [Jafferis, Pasquetti, Spiridonov-Vartanov,....]

Introduction

Our goal is to understand four-dimensional $\mathcal{N} = 1$ theories on four-manifolds $\mathcal{M}_4$.

We will also discuss three-dimensional $\mathcal{N} = 2$ theories on three manifolds $\mathcal{M}_3$.

Introduction

In general there is no unique way to compactify a given QFT on a curved manifold:

- We can add couplings to curvature.
- We can turn on additional background fields, such as various gauge fields that couple to currents etc.

The R-multiplet

The ambiguities in coupling the theory to curved space can be dealt with systematically.

We start from $\mathcal{N} = 1$ in $\mathbb{R}^4$. The energy-momentum tensor resides in a multiplet with bosonic components

$$\left(j_\mu^R, T_{\mu\nu}, F_{\mu\nu}\right) \ ,$$

where $\partial^\mu j_\mu^R = 0$, $dF = 0$, and the energy-momentum tensor is symmetric and conserved. This is a 12+12 multiplet.

[...,ZK-Seiberg, ...]

The R-multiplet

We couple the R-multiplet to background fields:

$T_{\mu\nu}$	$g_{\mu\nu}$
j_μ^R	A_μ^R
$F_{\mu\nu}$	$\epsilon_{\mu\nu\rho\sigma}B^{\rho\sigma}$

We emphasize again that the fields $g_{\mu\nu}, A_\mu^R, B_{\mu\nu}$ are not path integrated over.

For a general choice of these background fields, the coupling to curved space breaks supersymmetry.

The R-multiplet

It turns out that a necessary and sufficient condition to preserve at least one supercharge is that $\mathcal{M}_4$ is complex and the metric is Hermitian:

$$ds^2 = g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}}$$

The algebra satisfied by the single supercharge is $\delta^2 = 0$.

Another curious case is when the manifold is complex and looks locally like $\mathbb{T}^2 \times \Sigma$ where Σ is a Riemann surface. Then, one can preserve two supercharges.

From this point of view, $\mathbb{S}^3 \times \mathbb{S}^1$ is a (Hopf) fibration of $\mathbb{T}^2$ over $\mathbb{S}^2$ and thus clearly preserves two supercharges, but actually, this (and only this) special fibration preserves 4 supercharges.

[Festuccia-Seiberg, Dumitrescu-Festuccia-Seiberg]

Global Symmetries

Suppose we consider a theory that has a global symmetry group G. The bosonic operators in the supersymmetric current multiplet are

$$(J^G, j_\mu^G)\ .$$

We can couple them to background fields,

j_μ^G	A_μ^G
J^G	D^G

Supersymmetry is broken unless the connection A_μ^G is such that the G-bundle is holomorphic.

Global Symmetries

Hence, we have a leftover supersymmetry if and only if the manifold is complex and the global symmetry G-bundle is holomorphic.

The Partition Function

Let us specify a four-manifold $\mathcal{M}_4$ with some complex structure $J^2 = -1$ and a Hermitian metric $g_{i\bar{j}}$. If there are global symmetries, we can also specify holomorphic G-bundles. So we have

$$Z_{\mathcal{M}_4}[J_i^j, \overline{J}_{\bar{i}}^{\bar{j}}, g_{i\bar{j}}, A_\mu^G, ...]$$

The $\cdots$ stand for several additional parameters which enter the partition function, including the parameters of the Lagrangian itself.

The Partition Function

Our main concern is to understand what the partition function depends on, including the parameters parameterizing the geometry and the Lagrangian itself.

The Partition Function

Several general properties of the partition function $Z_{\mathcal{M}_4}[J_i^j, \overline{J}_{\bar{i}}^{\bar{j}}, g_{i\bar{j}}, A_\mu^G, \ldots]$:

- Given the complex structure $J^2 = -1$, the partition function is independent of the Hermitian metric $g_{i\bar{j}}$.
- The dependence on the complex structure is only through $\delta \overline{J}$, hence, it depends on the complex structure moduli holomorphically.
- The partition function depends holomorphically on the moduli of the holomorphic G-bundle.

The Partition Function

Therefore, the partition function of $\mathcal{N} = 1$ theories computes invariants of the complex structure and of the complex structure of G-bundles.

The Partition Function

Example: $\mathbb{S}^3 \times \mathbb{S}^1$ was studied by Kodaira-Spencer. Its moduli space of complex structure is two complex dimensional, with two natural coordinates introduced by Kodaira-Spencer, s, t.

On the other hand, physicists [Romelsberger...] computed the partition function of $\mathcal{N} = 1$ theories on $\mathbb{S}^3 \times \mathbb{S}^1$ and it was found that one can introduce two chemical potentials along the $\mathbb{S}^1$, denoted p, q. Those are in one-to-one correspondence with s, t of Kodaira-Spencer, and *the correspondence can be established explicitly*. A similar story for Lens spaces...

If we have a $U(1)$ global symmetry, the chemical potential z along the $\mathbb{S}^1$ corresponds to the modulus of a holomorphic $U(1)$ bundle over $\mathbb{S}^3 \times \mathbb{S}^1$.

The Partition Function with Two Supercharges

Consider complex manifolds which can be viewed as $\mathbb{T}^2$ fibration over a Riemann surface Σ. Such compactifications preserve two supercharges. The superalgebra is $\{\delta, \tilde{\delta}\} = \delta_K$ where K is a holomorphic Killing vector along the $\mathbb{T}^2$.

Loosely speaking, the theory on Σ is some deformation of the A-model. One can prove a stronger result than with one supercharge:

- The following components do not affect the partition function: $\delta \overline{J}^{\Sigma\Sigma}, \delta \overline{J}^{\Sigma\mathbb{T}^2}$. The partition function is only affected by $\delta \overline{J}^{\mathbb{T}^2\Sigma}, \delta \overline{J}^{\mathbb{T}^2\mathbb{T}^2}$.

Therefore, the complex structure moduli of the Riemann surface Σ do not affect the partition function.

Three-Dimensional $\mathcal{N} = 2$ Theories

One can similarly consider three-dimensional theories with four supercharges on $\mathbb{R}^3$ with an R-symmetry. One can ask which three-manifolds $\mathcal{M}_3$ are consistent with having a leftover supersymmetry.

The technical answer is as follows: Supersymmetry is preserved on almost contact manifolds $\mathcal{M}_3$ which obey an additional integrability condition. In detail, almost contact manifolds have $\eta_\mu, \xi^\mu, \Phi^\mu_\nu$ satisfying

$$\eta_\mu \xi^\mu = 1 \ , \qquad \Phi^\mu{}_\rho \Phi^\rho{}_\nu = -\delta^\mu_\nu + \xi^\mu \eta_\nu \ .$$

and the integrability condition reads

$$\Phi^\mu{}_\nu \mathcal{L}_\eta \Phi^\nu{}_\rho = 0 \ .$$

[Closset-Dumitrescu-Festuccia-ZK]

Three-Dimensional $\mathcal{N} = 2$ Theories

It appears that this integrability condition was not studied in the mathematical literature. Once we impose it on almost contact three-manifolds, one finds properties tantalizingly similar to complex geometry. There is an analog of Dolbeault cohomology and of Kodaira-Spencer theory. We analyzed basic aspects of the moduli space of integrable almost contact structure on $\mathbb{S}^3$.

We do not have a detailed understanding of which three-manifolds admit this new structure. It definitely exists on Seifert manifolds.

Three-Dimensional $\mathcal{N} = 2$ Theories

The supersymmetric partition function on three-manifolds is only sensitive to the integrable almost contact structure and not to the metric.

On manifolds with the differentiable structure of $\mathbb{S}^3$ there is a one-dimensional space of integrable almost contact structure. This is in correspondence with the squashing parameter denoted b in the literature.

Three-Dimensional $\mathcal{N} = 2$ Theories

Loosely speaking, no matter which new squashing of $\mathbb{S}^3$ is found, there will be no new partition functions beyond the one of Hama-Hosomichi-Lee. This is because the moduli space of integrable almost contact structures is one dimensional.

Actually, Hama-Hosomichi-Lee had two types of squashings, and our result clearly explains why one of them is trivial. (It has the same integrable almost contact structure as the round $\mathbb{S}^3$.)

Three-Dimensional $\mathcal{N} = 2$ Theories

Incidentally, this also explains why Imamura found the same answer as Hama-Hosomichi-Lee. More recently, new compactifications by Martelli-Passias appeared. From the considerations above, we predict they would yield the same partition functions as those of Hama-Hosomichi-Lee. This proposal is not inconsistent with their holographic results.

Three-Dimensional $\mathcal{N} = 2$ Theories

The dependence on the integrable almost contact structure may provide exact results about non-chiral correlation functions in flat space. For example in [Closset, Dumitrescu, ZK, Seiberg] we have shown how to compute various non-chiral two-point functions in $\mathbb{R}^3$ and used this as tests of dualities and RG flows

Can these results be derived in flat space directly?

Three-Dimensional $\mathcal{N} = 2$ Theories

Note that although the mathematical structure we encounter in 3d seems unfamiliar, it has several interesting properties, and it arises from physics as naturally as complex geometry in 4d.

Conclusions and Open Questions

- We used the tools of generating functionals and studied the partition function as a function of the topology, complex structure, and metric (and several other technical ingredients).
- We found that $4D$ $\mathcal{N}=1$ theories compute invariants of the complex structure, i.e. the partition function depends only on the underlying complex structure.
- $3D$ $\mathcal{N}=2$ theories compute invariants of a certain integrable almost contact structure.
- We discussed a new interpretation for the parameters appearing in the $\mathbb{S}^3 \times \mathbb{S}^1$ partition function (and Lens spaces). We also explained why the partition functions on various squashed three-spheres cannot have more information than the original formula of Hama-Hosomichi-Lee.

Conclusions and Open Questions

- Which additional conditions can be imposed on $\mathcal{M}_4$ such that the partition function becomes topological. If $\mathcal{M}_4$ is Calabi-Yau then one can argue that the answer is topological, but is it an exhaustive answer?
- The partition functions as functions of the complex structure moduli can have various poles. What is their meaning?
- Is there an obstruction to have an integrable almost contact structure on three-manifolds?
- When we have meromorphic functions (in this case of the complex structure moduli) we can often fix them by knowing something about the singularities and perhaps about global properties of the moduli space. Can we use these ideas to compute the partition function on any complex $\mathcal{M}_4$?
- Generalize to 6d, 5d, $\mathcal{N} = 2$ in 4d, etc.

Bonus Slides

Thanks for your Attention.

Bonus Slides

Bonus Slides

Relation to Twisting

We would like to clarify the relation with what is usually referred to as twisting.

Let us consider flat space for simplicity, $\mathbb{R}^4$. We choose a preferred spinor

$$\zeta_\alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This defines a complex structure on $\mathbb{R}^4$ via $J_{\mu\nu} = \zeta^\dagger \sigma_{\mu\nu} \zeta$. We denote the complex coordinates z, w.

Relation to Twisting

Holomorphic coordinate transformations induce $U(2) = SU(2) \times U(1)$ frame transformations of e_z, e_w. This is embedded in the usual tangent group $SU(2) \times SU(2)$. Thus we have a $U(1)$ which we can twist away with the R-symmetry $U(1)_R$.

The spinor $\zeta_\alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ becomes a scalar under *complex coordinate transformations*, but not under general coordinate transformations.

The resulting twisted theory has a scalar $\delta^2 = 0$, but the theory is not topological.

Outline of the Argument

Let us explain how some of the results explained here can be directly seen by concentrating on one patch with very small curvature. Then we can use the linearized analysis. One reconsiders the set of operators in the R-multiplet $\left(j_\mu^R, T_{\mu\nu}, F_{\mu\nu}\right)$.

One can ask the following question for δ defined by $\zeta_\alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

Which operators are δ closed and which of them are δ exact?

Outline of the Argument

It turns out that all δ-closed operators in the R-multiplet are δ-exact. In complex coordinates those are

$$T_{\bar{z}^{\bar{i}}\bar{z}^{\bar{j}}} + \cdots \ , T_{z^i,\bar{z}^{\bar{j}}} + \cdots$$

where the $\cdots$ are various other operators in the R-multiplet. This immediately means that the theory is independent of the Hermitian metric and it depends holomorphically on the complex structure, J^{z^i,z^j}.

Outline of the Argument

A puzzle: How can we change the complex structure consistently while preserving supersymmetry if T_{z^i,z^j} is not closed?

It turns out that there are operators of the type $T_{z^i,z^j} + \cdots$ which are not closed, but their δ variation produces total derivatives. Since complex structure deformations obey an integrability condition, this can be used to construct SUSY actions.

Similar considerations show that the partition function depends only on the actual change in the complex structure and not on the representative of it in the cohomology.

Outline of the Argument

The final step is to either argue that the linearized analysis in fact suffices, or to consider the generalization of the above ideas in the full non-linear theory.