3d&5d partition functions as q-CFT correlators

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based on arXiv:1303.2626 with F. Nieri and F. Passerini and work in progress

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- Ψ_0: field configurations satisfying localising (saddle point) equations
- with a clever localisation scheme, Ψ_0 is a finite dimensional set
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$\Rightarrow$ useful to study holography

$\Rightarrow$ connect to exactly solvable models such as 2d CFTs and TqFTs.

The S^4 localisation of 4d $\mathcal{N}=2$ theories [Pestun], has led to the AGT correspondence: [Alday-Gaiotto-Tachikawa],[Wyllard]

$$Z_{S^4}[\mathcal{T}_{g,n}] = \int [\mathrm{d}a]\ Z_{cl}\ Z_{1loop} \left| Z_{inst} \right|^2 = \int \mathrm{d}\alpha\ C \cdots C\ |\mathcal{F}_{\alpha}^{\alpha_i}(\zeta)|^2 = \langle \prod_i^n V_{\alpha_i} \rangle_{C_{g,n}}^{\mathrm{Liouville}}$$

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$$Z_{S_2}^{SQED} = \textstyle\sum_i^2 G_{cl}^{(i)} G_{1loop}^{(i)} \left| Z_V^{(i)} \right|^2 \sim \langle V_{\alpha_4} V_{\alpha_3}(1) V_{-b/2}(z) V_{\alpha_1} \rangle_{C_{0,4}}^{\mathrm{Liouville}}$$

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$$Z^{SQED}_{S_2} = \sum_i^2 G^{(i)}_{cl} G^{(i)}_{1loop} \left| Z^{(i)}_V \right|^2 \sim \langle V_{\alpha_4} V_{\alpha_3}(1) V_{-b/2}(z) V_{\alpha_1} \rangle^{\mathrm{Liouville}}_{C_{0,4}}$$

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Gauge theory and CFT are constrained by the same bootstrap equations!

Today I will argue that a similar story seems to hold in 3d and 5d:

- 3d partition functions $\Leftrightarrow$ degenerate q-CFT correlators
- 5d partition functions $\Leftrightarrow$ non-degenerate q-CFT correlators.

The starting point is the observation that partition functions of $\mathcal{N}=2$ theories on $S_b^3, S^2\times S^1$ can be expressed in terms of holomorphic blocks:

$$\mathcal{Z}[S_b^3, S^2\times S^1] := \mathcal{Z}_{S,id} = \sum_\alpha \left|\left|\mathcal{B}^\alpha(x,q)\right|\right|^2_{S,id}$$

[S.P.],[Beem-Dimofte-S.P.],[Hwang-Kim-Park],[Taki]

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Holomorphic blocks $\mathcal{B}^\alpha(x,q)$ are solid tori partition functions which

1. form a complete basis of solutions to certain difference operators
2. transform non-trivially under symmetries of partition functions
3. For $\mathcal{N}=2$ theories from M5 on a 3-manifold M, $\mathcal{B}^\alpha(x,q)$ are analytically continued Chern-Simons wavefunctions.
4. $\cdots$

On the basis of 1.+ 2. I will argue that there is a q-CFT structure.

Plan of the talk

- Block-factorisation of 3d partition functions
- q-CFT correlators via the bootstrap approach
- 3d and 5d partition functions as q-CFT correlators
- Conclusions and open issues

$\mathcal{N} = 2$ SQED on S_b^3

Consider the $\mathcal{N} = 2$ SQED, $U(1)$ gauge group, N_f chirals m_i, N_f anti-chirals $\tilde{m}_k$, with FI parameter ξ on

$$S_b^3: \quad b^2|z_1|^2 + \frac{1}{b^2}|z_2|^2 = 1$$

The Coulomb branch localisation yields:

$$Z_S^{SQED} = \int dx \; G_{cl} \cdot G_{1loop} = \int dx \; e^{2\pi i x \xi} \prod_{j,k}^{N_f} \frac{s_b(x + m_j + iQ/2)}{s_b(x + \tilde{m}_k - iQ/2)}$$

with

$$s_b(x) = \prod_{m,n \in \mathbb{Z}_{\geq 0}} \frac{mb + nb^{-1} + \frac{Q}{2} - ix}{mb + nb^{-1} + \frac{Q}{2} + ix}\,, \qquad Q = b + 1/b\,.$$

[Hama-Hosomichi-Lee]

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Flop Symmetry exchanges phase $I \leftrightarrow$phase II: $\xi \leftrightarrow -\xi$, $m_i \leftrightarrow -\tilde{m}_k$

Higgs-branch-like factorised form:

$$Z_S^{SQED} = \sum_i^{N_f} G_{cl}^{(i)} G_{1loop}^{(i)} \left|\left| \mathcal{Z}_V^{(i)} \right|\right|_S^2 = \sum_i^{N_f} \left|\left| \mathcal{B}^{(i)}(x, q) \right|\right|_S^2$$

[S.P.],[Beem-Dimofte-S.P.]

- $G_{cl}^{(i)}, G_{1loop}^{(i)}$ evaluated on SUSY vacua of the effective $(2,2)$ theory:

$$G_{cl}^{(i)} = e^{-2\pi i \xi m_i}, \qquad G_{1loop}^{(i)} = \prod_{j,k}^{N_f} \frac{s_b(m_j - m_i + iQ/2)}{s_b(\tilde{m}_k - m_i - iQ/2)},$$

- q-deformed vortices on $\mathbb{R}^2 \times S^1$:

$$\mathcal{Z}_V^{(i)} = \sum_n \prod_{j,k}^{N_f} \frac{(y_k x_i^{-1}; q)_n}{(q x_j x_i^{-1}; q)_n} z^n = {}_{N_f}\Phi_{N_f-1}^{(i)}(\vec{x}, \vec{y}; z).$$

- S-pairing: $\left|\left| f(x;q) \right|\right|_S^2 = f(x;q) f(\tilde{x}; \tilde{q})$

$$x_i = e^{2\pi b m_i}, \quad y_i = e^{2\pi b \tilde{m}_i}, \quad z = e^{2\pi b \xi}, \quad q = e^{2\pi i b^2},$$
$$\tilde{x}_i = e^{2\pi m_i/b}, \quad \tilde{y}_i = e^{2\pi \tilde{m}_i/b}, \quad \tilde{z} = e^{2\pi \xi/b}, \quad \tilde{q} = e^{2\pi i/b^2}$$

$\mathcal{N}=2$ SQED on $S^2\times S^1$

Consider the $\mathcal{N}=2$ SQED with fugacities:

$$
\begin{aligned}
&(\phi_i, r_i), \qquad i=1,\cdots N_f, \qquad \text{flavor} \quad U(1)^{N_f}\,,\\
&(\xi_i, l_i), \qquad i=1,\cdots N_f, \qquad \text{(anti)}-\text{flavor} \quad U(1)^{N_f}\,,\\
&(\omega, n), \qquad \text{topological} \quad U(1)\,,\\
&(t, s), \qquad \text{gauged} \quad U(1)\,.
\end{aligned}
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The Coulomb branch localisation yields:

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Z_{id}^{SQED}=\int dx\; G_{cl}\cdot G_{1loop}=\sum_{s\in\mathbb{Z}}\int\frac{dt}{2\pi i t}t^n\omega^s\prod_{j=1}^{N_f}\chi(t\phi_j, s+r_j)\prod_{k=1}^{N_f}\chi(\frac{1}{t\xi_k}, -s-l_k)
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with

$$
\chi(\zeta, m)=(q^{1/2}\zeta^{-1})^{-m/2}\prod_{l=0}^{\infty}\frac{(1-q^{l+1}\zeta^{-1}q^{-m/2})}{(1-q^{l}\zeta q^{-m/2})}
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[Kim],[Imamura-Yokoyama],[Kapustin-Willet],[Dimofte-Gukov-Gaiotto]

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[Beem-Dimofte-S.P.],[Dimofte-Gaitto-Gukov],[Hwang-Kim-Park]

- $G_{cl}^{(i)} G_{1loop}^{(i)}$ are evaluated on SUSY vacua:

$$G_{cl}^{(i)} = \omega^{-r_i}(\phi_i^{-1})^n, \quad G_{1loop}^{(i)} = \prod_{j=1}^{N_f} \chi(\phi_j\phi_i^{-1}, r_j - r_i) \prod_{k=1}^{N_f} \chi(\phi_i\xi_k^{-1}, r_i - l_k)\,,$$

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- *id*-pairing: $\left|\left|f(x;q)\right|\right|_{id}^2 := f(x;q) f(\tilde{x};\tilde{q})\,,$

$$x_i = \phi_i q^{r_i/2}, \qquad \tilde{x}_i = \phi_i^{-1} q^{r_i/2}\,, \qquad y_i = \xi_i q^{l_i/2}, \qquad \tilde{y}_i = \xi_i^{-1} q^{l_i/2}\,,$$

$$z = \omega q^{n/2}\,, \qquad \tilde{z} = \omega q^{-n/2}\,, \qquad \tilde{q} = q^{-1}$$

FLOP SYMMETRY is rather trivial on the Coulomb branch; but on the Higgs branch it implies non-trivial relations between blocks (analytic continuation $z \to z^{-1}$ from phases I to phase II):

$$Z^{I}_{id,S} = \sum_{i}^{N_f} G^{(i),I}_{cl} G^{(i),I}_{1loop} \left|\left| \mathcal{Z}^{(i),I}_{V} \right|\right|^2_{id,S} =$$

$$= \sum_{i}^{N_f} G^{(i),II}_{cl} G^{(i),II}_{1loop} \left|\left| \mathcal{Z}^{(i),II}_{V} \right|\right|^2_{id,S} = Z^{II}_{id,S}$$

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this structure is indeed reminiscent of 2d CFT correlators:

- gauge theory flop symmetry ⇔ crossing symmetry
- q-deformed hypers as chiral blocks ⇔ q-deformation of Virasoro +numerous "5d-AGT" results relating 5d instantons to q-Virasoro blocks. [Awata-Yamada],[many many others]

q-deformed Virasoro algebra $\mathcal{V}ir_{q,t}$

$\mathcal{V}ir_{q,t}$ has two complex parameters q, t and generators T_n with $n \in \mathbb{Z}$

[Shiraishi-Kubo-Awata-Odake],[Lukyanov-Pugai],[Frenkel-Reshetikhin],[Jimbo-Miwa]

$$[T_n\,,\,T_m] \;=\; -\sum_{l=1}^{+\infty} f_l\,(T_{n-l}T_{m+l} - T_{m-l}T_{n+l}) - \frac{(1-q)(1-t^{-1})}{1-p}((q/t)^n - (q/t)^{-n})\delta_{m+n,0}$$

where $f(z) = \sum_{l=0}^{+\infty} f_l z^l = \exp\left[\sum_{l=1}^{+\infty} \frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{1+(q/t)^n} z^n\right]$

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where $f(z) = \sum_{l=0}^{+\infty} f_l z^l = \exp\left[\sum_{l=1}^{+\infty} \frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{1+(q/t)^n} z^n\right]$

- For $t = q^{-b_0^2}$ and $q \to 1$, $\mathcal{V}ir_{q,t}$ reduces to Virasoro.
- chiral blocks with degenerate primaries (singular states in the Verma module) satisfy difference equations.

  [Awata-Kubo-Morita-Odake-Shiraishi], [Awata-Yamada],[Schiappa-Wyllard]

q-deformed Bootstrap Approach:

We will construct q-correlators using the conformal bootstrap approach:

3-point function is derived exploiting symmetries, without using the Lagrangian. [Belavin-Polyakov-Zamolodchikov],[Teschner]

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Consider 4-point function with a degenerate insertion

$$\langle V_{\alpha_4}(\infty) V_{\alpha_3}(r) V_{\alpha_2}(z,\tilde{z}) V_{\alpha_1}(0)\rangle \sim G(z,\tilde{z})$$

take $V_{\alpha_2}(z,\tilde{z})$ to have a null state at level 2, then

$$D(A,B;C;q;z)G(z,z)=0\,, \qquad D(\tilde{A},\tilde{B};\tilde{C};\tilde{q};\tilde{z})G(z,\tilde{z})=0\,,$$

where $D(A,B;C;q;z)$ is the q-hypergeometric operator.

$G(z,\tilde{z})$ is a bilinear combination of solutions of the q-hypergeometric eq.

Around $z = 0$

$$I_1^{(s)} = {}_2\Phi_1(A, B; C; z)\,, \qquad I_2^{(s)} = \frac{\theta(q^2 C^{-1} z^{-1}; q)}{\theta(qC^{-1}; q)\theta(qz^{-1}; q)}\, {}_2\Phi_1(qAC^{-1}, qBC^{-1}; q^2 C^{-1}; z)$$

For $q \to 1$ becomes the undeformed s-channel basis.

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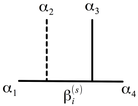
s-channel correlator:

$$\langle V_{\alpha_4}(\infty) V_{\alpha_3}(r) V_{\alpha_2}(z) V_{\alpha_1}(0)\rangle \sim \sum_{i,j=1}^{2} \tilde{I}_i^{(s)} K_{ij}^{(s)} I_j^{(s)}$$

$$= \sum_{i=1}^{2} K_{ii}^{(s)} \left|\left| I_i^{(s)} \right|\right|_*^2 = \sum_i$$



$K_{ij}^{(s)}$ is diagonal with elements related to 3-point functions

$$K_{ii}^{(s)} = C(\alpha_4, \alpha_3, \beta_i^{(s)})\, C(Q_0 - \beta_i^{(s)}, -b_0/2, \alpha_1)\,, \quad \beta_i^{(s)} = \alpha_1 \pm \frac{b_0}{2}\,, \;\; i = 1, 2$$

For the moment assume generic pairing $\left|\left|(\cdots)\right|\right|_*^2$.

Around $z = \infty$

$$I_1^{(u)} = \frac{\theta(qA^{-1}z^{-1};q)}{\theta(A^{-1};q)\theta(qz^{-1};q)}\ {}_2\Phi_1(A, qAC^{-1}; qAB^{-1}; q^2z^{-1}),$$

$$I_2^{(u)} = \frac{\theta(qB^{-1}z^{-1};q)}{\theta(B^{-1};q)\theta(qz^{-1};q)}\ {}_2\Phi_1(B, qBC^{-1}; qBA^{-1}; q^2z^{-1})$$

For $q \to 1$ limit becomes the undeformed u-channel basis.

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u-channel correlator:

$$\langle V_{\alpha_4}(\infty)V_{\alpha_3}(r)V_{\alpha_2}(z)V_{\alpha_1}(0)\rangle \sim \sum_{i,j=1}^{2} \tilde{I}_i^{(u)} K_{ij}^{(u)} I_j^{(s)}$$
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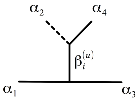

$K_{ij}^{(u)}$ is diagonal with elements related to 3-point functions

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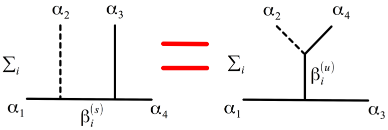
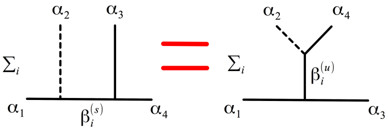
impose crossing symmetry



$$K_{11}^{(s)}\left|\left|I_1^{(s)}\right|\right|_*^2 + K_{22}^{(s)}\left|\left|I_2^{(s)}\right|\right|_*^2 = K_{11}^{(u)}\left|\left|I_1^{(u)}\right|\right|_*^2 + K_{22}^{(u)}\left|\left|I_2^{(u)}\right|\right|_*^2$$

impose crossing symmetry



$$K_{11}^{(s)} \left|\left| I_1^{(s)} \right|\right|_*^2 + K_{22}^{(s)} \left|\left| I_2^{(s)} \right|\right|_*^2 = K_{11}^{(u)} \left|\left| I_1^{(u)} \right|\right|_*^2 + K_{22}^{(u)} \left|\left| I_2^{(u)} \right|\right|_*^2$$

analytic continuation $I_i^{(s)} = \sum_{j=1}^2 M_{ij} I_j^{(u)}$, $\tilde{I}_i^{(s)} = \sum_{j=1}^2 \tilde{M}_{ij} \tilde{I}_j^{(u)}$ yields:

$$\sum_{k,l=1}^2 K_{kl}^{(s)} \tilde{M}_{ki} M_{lj} = K_{ij}^{(u)}$$

Solving these equations we can determine 3-point functions. But we need to specify the pairing $\left|\left|(\cdots)\right|\right|_*^2 \to$ use 3d gauge theory pairings!

id-pairing *q*-CFT

Now assume that chiral blocks are paired as:

$$\Big|\Big| f(x;q) \Big|\Big|^2_{id} = f(x;q) f(\tilde{x};\tilde{q})\,.$$

with:

$$x = e^{\beta X}\,, \quad \tilde{x} = e^{-\beta X}\,, \quad \tilde{q} = q^{-1}$$

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The bootstrap equations are solved by:

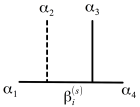
$$C_{id}(\alpha_3,\alpha_2,\alpha_1) = \frac{1}{\Upsilon^\beta(2\alpha_T - Q_0)} \prod_{i=1}^{3} \frac{\Upsilon^\beta(2\alpha_i)}{\Upsilon^\beta(2\alpha_T - 2\alpha_i)}$$

where $2\alpha_T = \alpha_1 + \alpha_2 + \alpha_3$, $Q_0 = b_0 + 1/b_0$ and

$$\Upsilon^\beta(X) \propto \prod_{n_1,n_2=0}^{\infty} \sinh\left[\frac{\beta}{2}\left(X + n_1 b_0 + \frac{n_2}{b_0}\right)\right] \sinh\left[\frac{\beta}{2}\left(-X + (n_1+1) b_0 + \frac{(n_2+1)}{b_0}\right)\right]$$

SQED $N_f = 2$ on $S^2 \times S^1 \Leftrightarrow$ *id*-pairing 4-point degenerate correlator

$$Z_{id}^{SQED} = \sum_{i=1}^{2} G_{cl}^{(i),I} G_{1loop}^{(i),I} \left|\left| \mathcal{Z}_V^{(i),I} \right|\right|_{id}^2 \sim \sum_{i=1}^{2} K_{ii}^{(s)} \left|\left| I_i^{(s)} \right|\right|_{id}^2 = \sum_i$$



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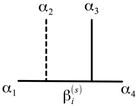

dictionary:

$$z_{CFT} \sim z_{gauge}\,, \quad q = e^{\beta/b_0}\,, \quad \alpha_2 = -b_0/2$$

$$\alpha_1 = \frac{Q_0}{2} + i\frac{\Phi_1 - \Phi_2}{2}\,, \quad \alpha_3 = \frac{b_0}{2} - i\frac{\Xi_1 + \Xi_2 - \Phi_1 - \Phi_2}{2}\,, \quad \alpha_4 = \frac{Q_0}{2} - i\frac{\Xi_1 - \Xi_2}{2}\,,$$

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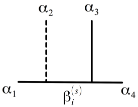

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- gauge theory flop symmetry $\Leftrightarrow$ q-CFT crossing symmetry
- $\beta \to 0$ limit recovers [Dorud-Gomis-LeFloch-Lee]
  - CFT: $\mathcal{V}ir_{q,t} \to$ Virasoro, we recover Liouville theory results
  - gauge: $S^2 \times S^1$ partition function reduces to S^2 partition function

S-pairing q-CFT

Now assume that chiral blocks are paired as:

$$\Big|\Big| f(x;q) \Big|\Big|_S^2 = f(x;q) f(\tilde{x};\tilde{q})\,.$$

where

$$x = e^{2\pi i X/\omega_2}\,, \quad \tilde{x} = e^{2\pi i X/\omega_1}, \quad q = e^{2\pi i \frac{\omega_1}{\omega_2}}\,, \quad \tilde{q} = e^{2\pi i \frac{\omega_2}{\omega_1}}$$

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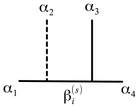
$$C_S(\alpha_3,\alpha_2,\alpha_1) = \frac{1}{S_3(2\alpha_T - E)}\prod_{i=1}^{3}\frac{S_3(2\alpha_i)}{S_3(2\alpha_T - 2\alpha_i)}$$

where $E = \omega_1 + \omega_2 + \omega_3$ and

$$S_3(X) \propto \prod_{n_1,n_2,n_3=0}(\omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 + X)\,(\omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 + E - X)$$

SQED $N_f = 2$ on $S_b^3 \Leftrightarrow S$-pairing 4-point degenerate correlator

$$Z_S^{SQED} = \sum_{i=1}^{2} G_{cl}^{(i),I} G_{1loop}^{(i),I} \left|\left| \mathcal{Z}_V^{(i),I} \right|\right|_S^2 \sim \sum_{i=1}^{2} K_{ii}^{(s)} \left|\left| I_i^{(s)} \right|\right|_S^2 = \sum_i$$



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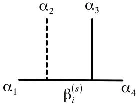

dictionary:

$$\alpha_2 = -\omega_3/2, \qquad \omega_1 = b, \qquad \omega_2 = \frac{1}{b}, \qquad z_{CFT} \sim z_{gauge}$$

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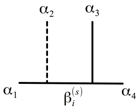

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- gauge theory flop symmetry $\Leftrightarrow$ q-CFT crossing symmetry
- three possibilities:

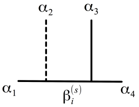
$$\alpha_2 = -\omega_k/2, \qquad b = \omega_i, \qquad \frac{1}{b} = \omega_j\,, \quad i \neq j \neq k = 1, 2, 3\,.$$

corresponding to the three big deformed S^3 inside a deformed S^5.

so far:

3d gauge theory partition functions $\Leftrightarrow$ q-CFT degenerate correlators

$$Z^{SQED}_{S,id} = \sum_{i=1}^{2} G_{cl}^{(i),I} G_{1loop}^{(i),I} \left|\left| Z_V^{(i),I} \right|\right|^2_{S,id} \sim \sum_{i=1}^{2} K_{ii}^{(s)} \left|\left| I_i^{(s)} \right|\right|^2_{S,id} = \sum_i$$



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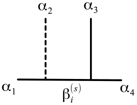
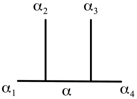
$$Z^{SQED}_{S,id} = \sum_{i=1}^{2} G^{(i),I}_{cl}\, G^{(i),I}_{1loop} \left|\left| Z^{(i),I}_{V} \right|\right|^2_{S,id} \sim \sum_{i=1}^{2} K^{(s)}_{ii} \left|\left| I^{(s)}_{i} \right|\right|^2_{S,id} = \sum_{i}$$



Let's now consider non-degenerate correlators

Example:

$$\langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle_{S,id} = \int d\alpha$$



$$= \int d\alpha\; C_{S,id}\; C_{S,id}\; (\mathrm{Conf.Blocks})$$

in analogy with the AGT case, one could expect that

5d gauge theory partition functions $\Leftrightarrow$ q-CFT non-degenerate correlators

$\mathcal{N} = 1$ theories on $S^4 \times S^1$

Computes the super-conformal index:

$$I_5 = \mathrm{Tr}\ (-1)^F q^{J_{12}-R} t^{J_{34}-R} z_j^{f_j}$$

Coulomb branch localisation yields:

$$Z_{S^4 \times S^1} = \int d\vec{\sigma}\ \mathcal{Z}_{1\text{loop}}(\vec{\sigma}, \vec{m}) \left| Z_{inst}^{5d}(\vec{\sigma}, \vec{m}, z) \right|^2$$

[Kim-Kim-Lee],[Terashima],[Iqbal-Vafa]

- $\left| Z_{inst}^{5d}(\vec{\sigma}, \vec{m}, z) \right|^2$ comes from point-like instantons at N and S poles

- 1loop contribution can be re-written as:
  - vector multiplet:

$$\mathcal{Z}_{1\text{loop}}^{\text{vect}}(\sigma) = \prod_{\alpha>0} \Upsilon^\beta \left(i\alpha(\sigma)\right) \Upsilon^\beta \left(-i\alpha(\sigma)\right)$$

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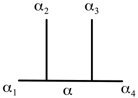
$$Z^{SCQCD}_{S^4\times S^1} = \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4}\rangle_{id} = \int d\alpha$$



- 5d instantons vs $\mathcal{V}ir_{qt}$ non-degenerate conformal blocks:
  [Awata-Yamada],[Mironov-Morozov-Shakirov-Smirnov]

$$Z^{5d,SCQCD}_{inst} = \mathcal{F}^{qt}_{\alpha_1\alpha_2\alpha\alpha_3\alpha_4}(z)$$

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$$\mathcal{Z}^{\text{vect}}_{\text{1loop}}(\sigma)\prod_{i=1}^{4}\mathcal{Z}^{\text{hyper}}_{\text{1loop}}(\sigma,m_i) = C_{id}(\alpha_1,\alpha_2,\alpha)C_{id}(Q_0-\alpha,\alpha_3,\alpha_4)$$

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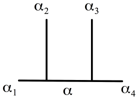

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$\mathcal{N} = 1$ theories on S^5

(see Seok Kim's talk)

Localisation on $\omega_1^2|z_1|^2 + \omega_2^2|z_2|^2 + \omega_3^2|z_3|^2 = 1$ yields:

$$Z_{S^5} = \int d\vec{\sigma}\ Z_{cl}(\vec{\sigma}, \tau; \vec{\omega})\ Z_{1loop}(\vec{\sigma}, \vec{m}; \vec{\omega})$$

$$\times Z_{inst}^{5d,I}(\vec{\sigma}, \vec{m}, z; q, t)\ Z_{inst}^{5d,II}(\vec{\sigma}, \vec{m}, z; q, t)\ Z_{inst}^{5d,III}(\vec{\sigma}, \vec{m}, z; q, t)^{-1}$$

[Lockhart-Vafa],[Kim-Kim-Kim]

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[Lockhart-Vafa],[Kim-Kim-Kim]

- Instantons comes with equivariant parameters

$$(q, t):\ \left(e^{2\pi i \frac{\omega_2}{\omega_1}}, e^{-2\pi i \frac{\omega_3}{\omega_1}}\right)_I,\ \left(e^{2\pi i \frac{\omega_1}{\omega_2}}, e^{-2\pi i \frac{\omega_3}{\omega_2}}\right)_{II},\ \left(e^{2\pi i \frac{\omega_1}{\omega_3}}, e^{2\pi i \frac{\omega_2}{\omega_3}}\right)_{III}$$

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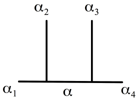

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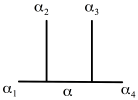

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Test:

For $\alpha_2 \to -\omega_3/2$ the S^5 partition function degenerates:

$$\boxed{Z_{S^5}^{SCQCD} \to Z_{S_b^3}^{SQED}}$$

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For $\alpha_2 \to -\omega_3/2$ the S^5 partition function degenerates:

$$\boxed{Z_{S^5}^{SCQCD} \to Z_{S_b^3}^{SQED}}$$

Because of zeros and poles in Z_{1loop} the integration contour is pinched and gets contributions only from $\alpha = \alpha_1 \pm \omega_3/2$:

$$\int d\alpha \; Z_{1loop} \; Z_{cl} \; Z_{inst}^{5d,I} Z_{inst}^{5d,II} Z_{inst}^{5d,III} \to \sum_{i=1}^{2} G_{cl}^{(i)} G_{1loop}^{(i)} \left|\left| \mathcal{Z}_V^{(i)} \right|\right|_S^2$$

$$Z_{inst}^{5d,I} = \sum_{Y_1,Y_2} (\cdots) \to \sum_{0,1^n} (\cdots) = \mathcal{Z}_V^{(1,2)}, \qquad Z_{inst}^{5d,II} = \sum_{W_1,W_2} (\cdots) \to \sum_{0,n} (\cdots) = \tilde{\mathcal{Z}}_V^{(1,2)},$$

$$Z_{inst}^{5d,III} = \sum_{X_1,X_2} (\cdots) \to \sum_{0,0} (\cdots) = 1$$

and likewise for permutations of $\omega_1, \omega_2, \omega_3$.

Test:

For $\alpha_2 \to -\omega_3/2$ the S^5 partition function degenerates:

$$\boxed{Z_{S^5}^{SCQCD} \to Z_{S_b^3}^{SQED}}$$

Because of zeros and poles in Z_{1loop} the integration contour is pinched and gets contributions only from $\alpha = \alpha_1 \pm \omega_3/2$:

$$\int d\alpha \; Z_{1loop} \; Z_{cl} \; Z_{inst}^{5d,I} Z_{inst}^{5d,II} Z_{inst}^{5d,III} \to \sum_{i=1}^{2} G_{cl}^{(i)} G_{1loop}^{(i)} \left|\left| \mathcal{Z}_V^{(i)} \right|\right|_S^2$$

$$Z_{inst}^{5d,I} = \sum_{Y_1,Y_2} (\cdots) \to \sum_{0,1^n} (\cdots) = \mathcal{Z}_V^{(1,2)}, \qquad Z_{inst}^{5d,II} = \sum_{W_1,W_2} (\cdots) \to \sum_{0,n} (\cdots) = \tilde{\mathcal{Z}}_V^{(1,2)},$$

$$Z_{inst}^{5d,III} = \sum_{X_1,X_2} (\cdots) \to \sum_{0,0} (\cdots) = 1$$

and likewise for permutations of $\omega_1, \omega_2, \omega_3$.

- Similar story for the $S^3 \times S^1$ index: for special values of fugacities the index satisfies difference equations. [Gaiotto-Rastelli-Razamat]
- Degenerate correlators/$\mathcal{Z}_{S_b^3}^{SQED}$ are crossing-symmetry/flop invariant;

  Is there crossing-symmetry on S^5, what is its gauge theory meaning?

Conclusions & outlook

Hints of a q-CFT-like structure in 5d and 3d partition functions.

- Study modular invariance in the 5d/non-degenerate case.
- Consider other pairings $\left|\left|(\cdots)\right|\right|_*^2$ and other geometries.
- Use q-CFT to study gauge theory. For example construct q-CFT Verlinde loop operators and study their gauge theory meaning.