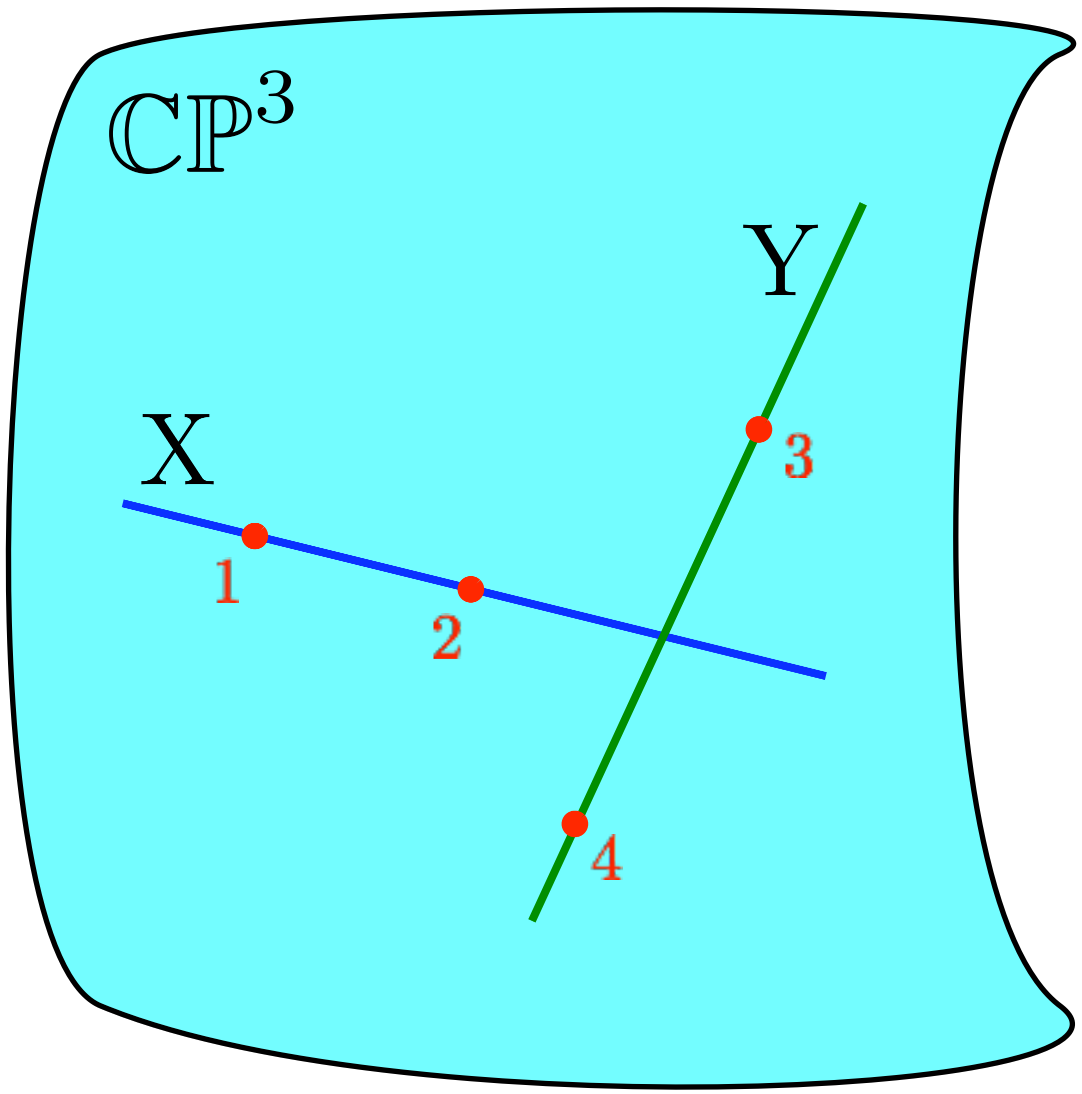
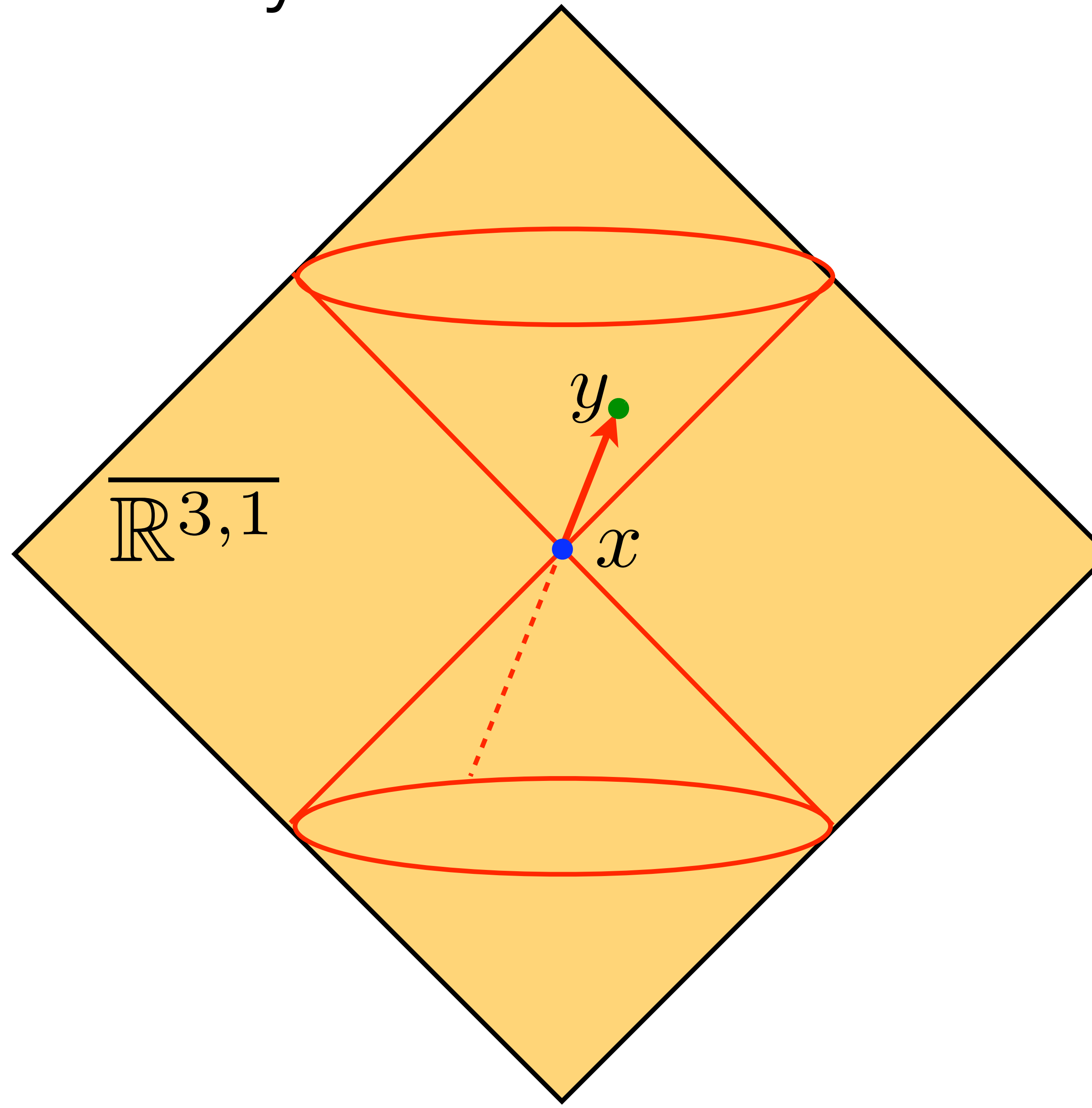
Twistor strings for $\mathcal{N} = 8$ supergravity

David Skinner - IAS & Cambridge

Strings 2013 - Sogang University

Twistor space is $\mathbb{CP}^3$, described by co-ords $Z^a \sim r Z^a$





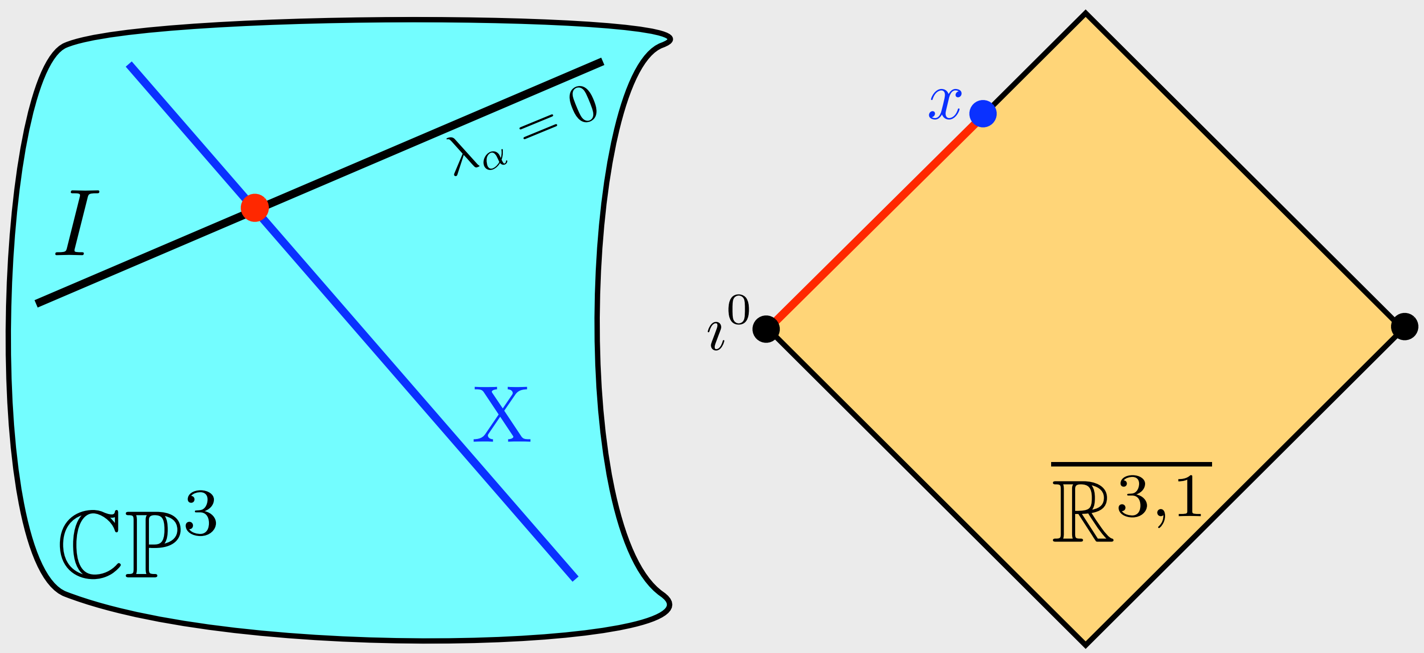
$\mathbb{CP}^1$ in twistor space ⟷ Point in space-time

Two lines intersect ⟷ Separation is null

$$\mathrm{X}^{ab} = Z_1^{[a} Z_2^{b]} \qquad \mathrm{Y}^{cd} = Z_3^{[c} Z_4^{d]} \qquad \epsilon(1,2,3,4) \propto (x-y)^2$$

To define a metric, not just a conformal structure, we must also choose an ***infinity twistor*** $I^{ab} = I^{[ab]}$

For flat space-time the infinity twistor represents a line. In terms of the coords



$$Z^a = (\mu^{\dot\alpha}, \lambda_\alpha),$$

$$I^{ab} = \begin{pmatrix} \epsilon^{\dot\alpha\dot\beta} & 0 \\ 0 & 0 \end{pmatrix}$$

and is the line $\lambda_\alpha = 0$

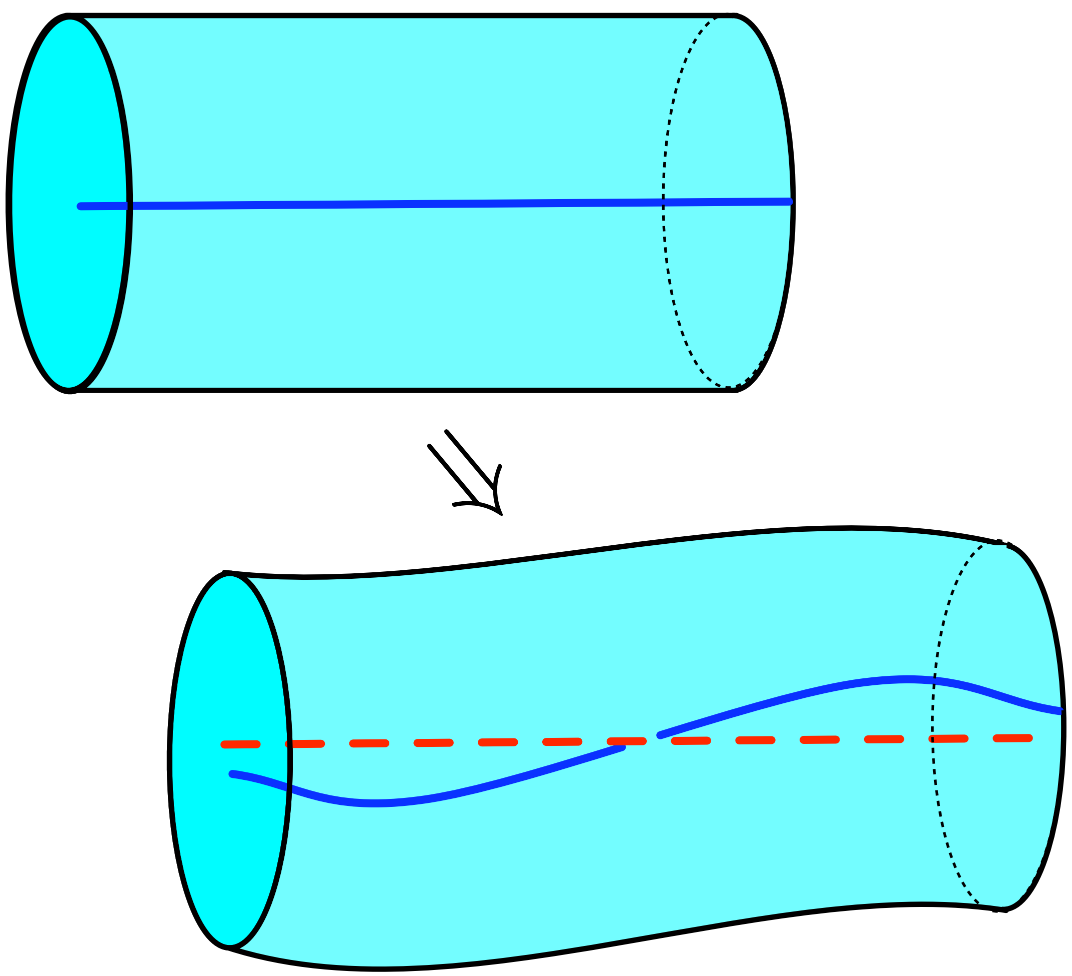
I breaks conformal invariance and sets a ***mass scale***

$$(x-y)^2 = \frac{\epsilon(1,2,3,4)}{\langle 12 \rangle \langle 34 \rangle} \qquad \langle ij \rangle := \epsilon_{abcd} I^{ab} Z^c_{(i)} Z^d_{(j)}$$

To describe (self-dual) gravity, we deform the $\mathbb{C}$-structure

$$\bar{\partial} \longrightarrow \bar{\partial} + V \qquad V \in H^{0,1}(\mathbb{PT}, T_{\mathbb{PT}})$$

[Penrose; Ward; Atiyah,Hitchin,Singer]



Arbitrary deformations give s.d. ***conformal*** gravity. To yield a vacuum Einstein metric, V must be Hamiltonian

$$V = \{h, \ \} = I^{ab} \frac{\partial h}{\partial Z^a} \frac{\partial}{\partial Z^b}$$

w.r.t. the Poisson bracket defined by the infinity twistor

$h \in H^{0,1}(\mathbb{PT}, \mathcal{O}(2))$ is the twistor space wavefunction of a positive helicity graviton. Extends to $\mathcal{N} = 8$ multiplet

$$h(Z, \chi) = h(Z) + \chi^A \psi_A(Z) + \cdots + (\chi)^8 \tilde{h}(Z)$$

The infinity twistor is also important in governing the structure of scattering amplitudes

When written on twistor space, the n-particle, g-loop amplitude with $n_\pm$ external gravitons of helicity ± 2 is a monomial with

$n_+ + g - 1$ powers of $I^{ab} \leftrightarrow [\ ,\ ]$

and

$n_- + g - 1$ powers of $I_{ab} \leftrightarrow \langle\ ,\ \rangle$

(parity)

- g-loop, n-pt Feynman diagram $\propto \kappa^{n+2g-2}$. In twistor space, each κ is accompanied by an infinity twistor
- parity exchanges $[\ ,\ ]$ with $\langle\ ,\ \rangle$
- conformal breaking is made explicit

All ***MHV*** tree amplitudes in $\mathcal{N}=8$ sugra are given by

$$\mathcal{M}_n^{\rm MHV} = \delta^{4|16}\left(\sum_{i=1}^{n} p_i\right) \frac{\|\mathrm{H}\|^{ijk}_{rst}}{\langle ij\rangle\langle jk\rangle\langle ki\rangle\;\langle rs\rangle\langle st\rangle\langle tr\rangle}$$

on momentum space, where H is the symmetric matrix

$$\mathrm{H}_{ij} = \frac{[ij]}{\langle ij\rangle} \qquad \mathrm{H}_{ii} = -\sum_{j\neq i} \mathrm{H}_{ij} \frac{\langle pj\rangle\langle qj\rangle}{\langle p\,i\rangle\langle q\,i\rangle}$$

[Hodges]

Permutation symmetric ***without explicit sum!***

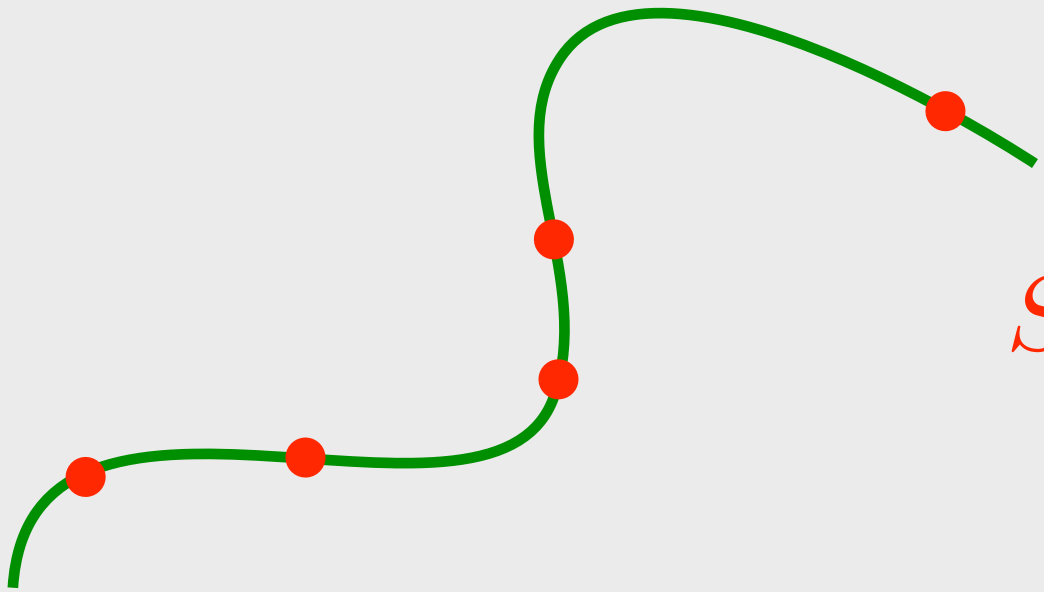
- determinant suggests correlator of fermion bilinears

$\mathrm{rk}(\mathrm{H}) = (n-3)$ and $\|\mathrm{H}\|^{ijk}_{rst}$ is an $(n-3)$ minor

- provides required number $(n_+ - 1)$ of $[\;,\;]$ brackets
- suggests fixing of some residual fermionic symmetry

The worldsheet theory

Like the Berkovits - Witten twistor string, the model is based on holomorphic maps to twistor space, here with $\mathcal{N} = 8$ supersymmetry

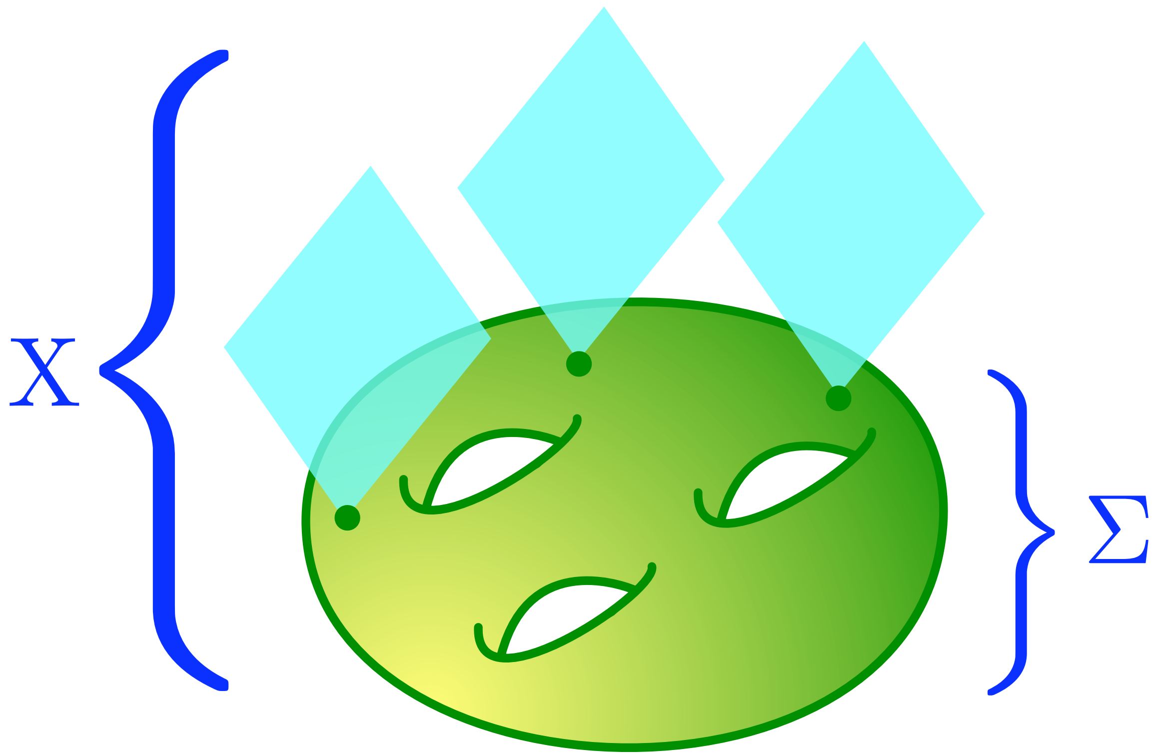


$$S = \int_\Sigma Y_I(\bar\partial + \bar A)Z^I + \cdots$$

Additional fields needed to:

- introduce dependence on infinity twistor
- provide worldsheet version of Hodges' matrix
- cancel anomalies ($\mathbb{CP}^{3|8}$ is not sCY)

Extend Σ to a 1|2-dimensional supermanifold $\mathrm{X} \to \Sigma$, described locally by coords (x, θ^a)



Vectors $\mathcal{V}^a(x,\theta)\frac{\partial}{\partial\theta^a}$ in fermionic directions obey $\mathfrak{sl}(1|2)$ algebra

- four bosonic & four fermionic generators
- maximal bosonic subalgebra $\mathfrak{gl}(2)_R \cong \mathfrak{gl}(1) \oplus \mathfrak{sl}(2)$

twist by $\mathfrak{gl}(1)$ scaling of target

θ^a have conformal weight -½ (as in RNS) & charge +1

The matter & ghost fields are

$$\mathcal{Z}^I(x,\theta) = Z^I(x) + \theta^a \rho_a^I(x) + \theta^2 Y^I(x)$$

$$C^a(x,\theta) = \gamma^a(x) + \theta^b N^a_b(x) + \theta^2 \nu^a(x)$$

$$B_a(x,\theta) = \mu^a(x) + \theta^b M_{ab}(x) + \theta^2 \beta_a(x)$$

In the gauge $\bar{A}_{\mathfrak{sl}(1|2)} = 0$, the worldsheet action is

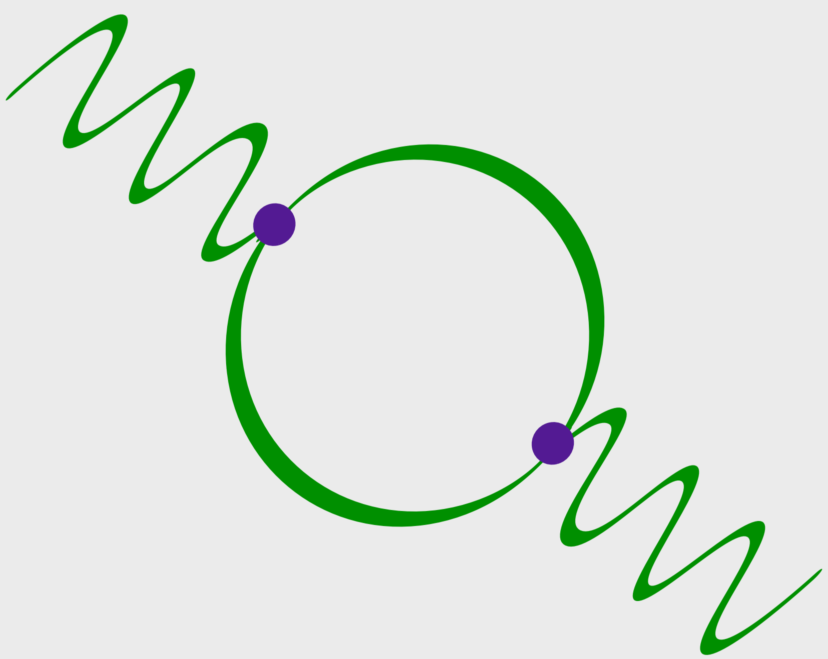
$$S = \int_X \mathrm{d}^{1|2}x \, \langle \mathcal{Z}, \bar{\partial}\mathcal{Z}\rangle + B_a \bar{\partial} C^a$$

while the (classically) nilpotent BRST operator is

$$Q = \oint \mathrm{d}^{1|2}x \, \langle \mathcal{Z}, C^a \partial_a \mathcal{Z}\rangle - \frac{1}{2} B_a [C,C]^a$$

- BRST operator depends on the infinity twistor $\langle \, , \rangle$ breaking conformal invariance

Gauge anomalies cancel iff twistor space has $\mathcal{N}=8$ supersymmetry



GL(1) anomaly:

$$\sum_i (-1)^{F_i} q_i^2 = (\underset{YZ}{4-\mathcal{N}}) + \underset{\beta\gamma}{2} + \underset{\mu\nu}{2}$$

SL(2) anomaly:

$$\sum_i \frac{(-1)^{F_i}}{|\mathrm{Aut}\Gamma_i|}\,\mathrm{tr}_{R_i}(t\cdot t) = \frac{3}{4}(\mathcal{N}-8)$$

- involves both ghosts and matter; cancellation not solely due to supersymmetry of target space

Positively charged fields have zero modes:

$\mathcal{Z}^I : \; d+1-g$

selection rule relating
MHV level to degree of curve
$n_- = d+1-g$

$\gamma^a : \; d+2-2g$

zero modes of bosonic ghost -
fix residual fermionic symmetry
$\#\gamma_{\rm zm} = n - \#[\;,\;]$

$\mu_a : \; d$

zero modes of bosonic antighost -
fermionic moduli (handle by PCOs)
$\#\mu_{\rm zm} = \#\langle\;,\;\rangle$

Path integral measure over all z.m. has no net charge

The total Virasoro central charge is

$$c = \underset{YZ}{2(4-\mathcal{N})} + \underset{\bar{\rho}\rho}{(4-\mathcal{N})} + \underset{\beta\gamma}{22} - \underset{MN}{8} - \underset{\mu\nu}{2} = 3(8-\mathcal{N})$$

so also vanishes with $\mathcal{N} = 8$ twistor target space (as do mixed Virasoro / gauge anomalies)

The worldsheet theory is thus some $c = 0$ CFT

- holomorphic, but not a TQFT. $T \neq \{Q, \cdot\}$
- include bc ghosts and some other "internal" CFT with $c = 26$. Not important at tree level, but presumably crucial for higher genus.

Matter vertex operators are similar to RNS string:

$$c\delta^2(\gamma)h(Z) \qquad \text{or} \qquad U \equiv \int_\Sigma \delta^2(\gamma)\, h(Z)$$

for 'fixed' vertex operators. Integrated operators are

$$V \equiv \int \mathrm{d}^2\theta\, h(\mathcal{Z}) = \int_\Sigma \left[\frac{\partial h}{\partial Z}, Y\right] - \rho^I \frac{\partial}{\partial Z^I}\left[\bar{\rho}, \frac{\partial h}{\partial Z}\right]$$

describing deformations of the worldsheet action

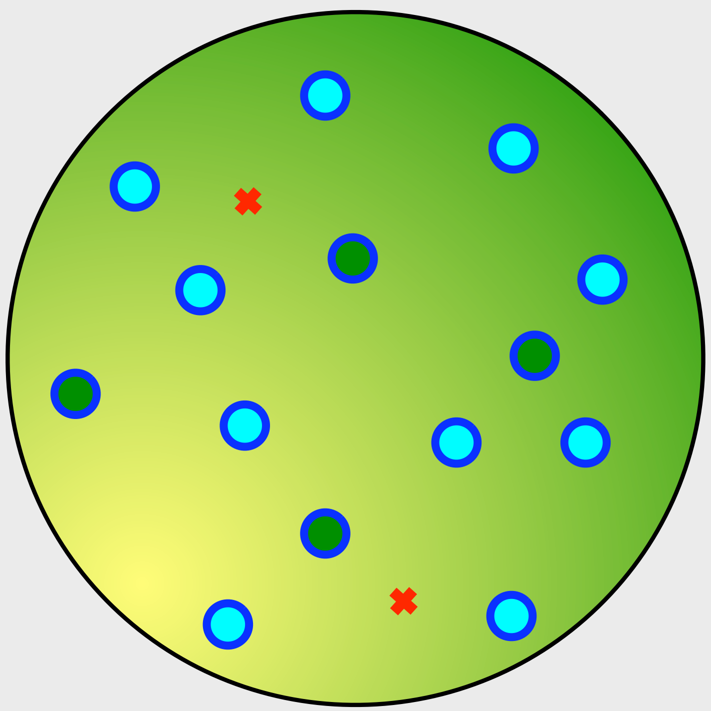
- h is the twistor wavefunction of an $\mathcal{N} = 8$ graviton

Picture changing operators (associated to μ zm) are

$$\Upsilon \equiv \prod_{a=1,2} [Q, \Theta(\mu_a)] = \delta^2(\mu)\langle\rho, Z\rangle\, \bar{\rho}_I Z^I + \cdots$$

All tree-level amplitudes in $\mathcal{N}=8$ supergravity come from the $g=0$ twistor string correlator

$[\;,\;]$ dependence lives here



$$\left\langle cU_1\, cU_2\, cU_3 \prod_{i=4}^{d+2}\int U_i \prod_{j=d+3}^{n}\int V_j \prod_{k=1}^{d}\Upsilon \right\rangle$$

$\langle\;,\;\rangle$ dependence from here

- ‘fixed’ vertex op
- integrated vertex op
- PCO

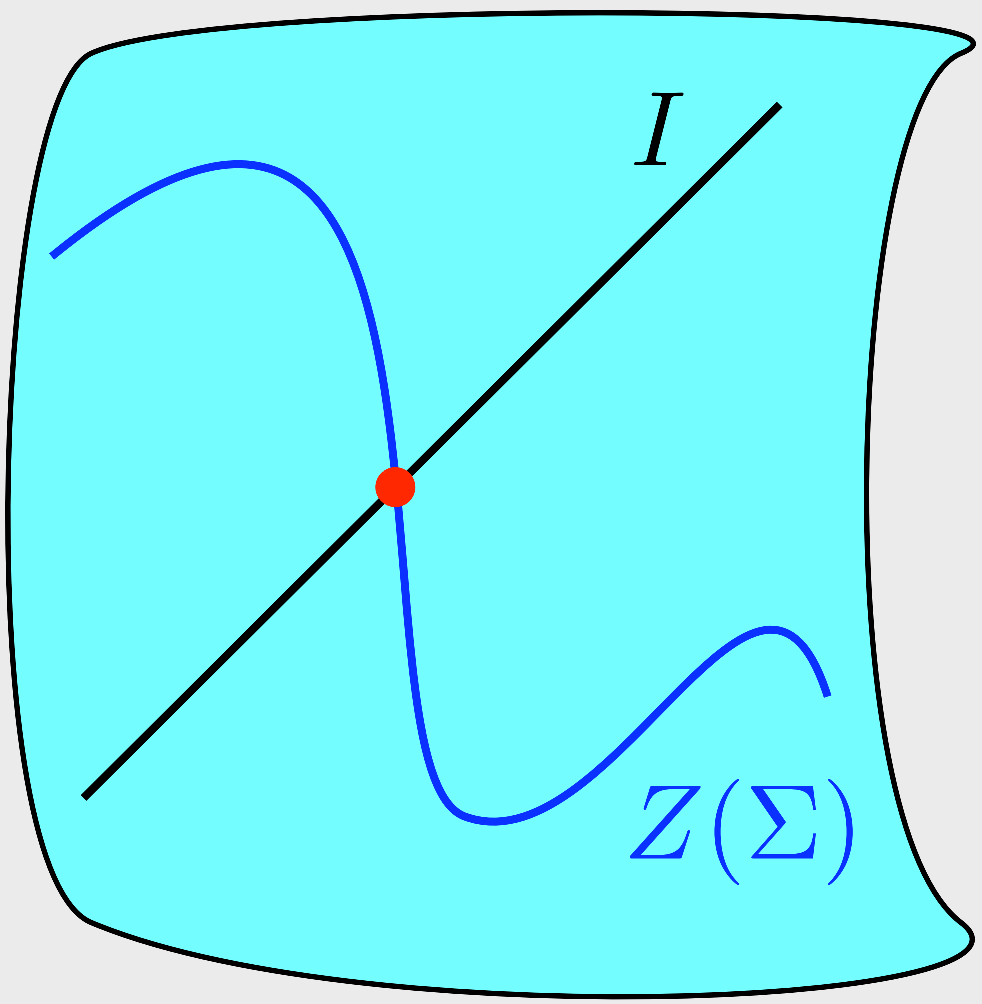
Correlator of PCOs is independent of insertion points

$$\left\langle \prod_{k=1}^{d} \Upsilon(x_k) \right\rangle = \mathrm{R}(\lambda_\alpha)$$

$\Upsilon \leftarrow \delta^2(\mu)\langle\rho Z\rangle\bar{\rho}Z$

the ***resultant*** of the two λ_α components of $Z : \Sigma \to \mathbb{CP}^{3|8}$

[Cachazo]



$\mathrm{R}(\lambda_\alpha) = 0 \longleftrightarrow \lambda_\alpha(x_*) = 0$ for some $x_* \in \Sigma$

$\lambda_\alpha = 0$ is the line I at infinity

The amplitude thus lives on holomorphic curves in $\mathbb{CP}^{3|8} - I$, the 'inside' of space-time

[Casali,DS; Cachazo,He,Yuan]

The remaining correlator of matter vertex operators

$$\left\langle \prod_{k=1}^{d+2} \delta^2(\gamma) h_k \prod_{\ell=d+3}^{n} \left(\left[Y, \frac{\partial h_\ell}{\partial Z} \right] + \rho \frac{\partial}{\partial Z} \left[\bar{\rho}, \frac{\partial h_\ell}{\partial Z} \right] \right) \right\rangle = \frac{\|\Phi\|^{r_1 \cdots r_{d+2}}_{c_1 \cdots c_{d+2}}}{\|\omega_j(x_{r_k})\| \; \|\omega_l(x_{c_m})\|}$$

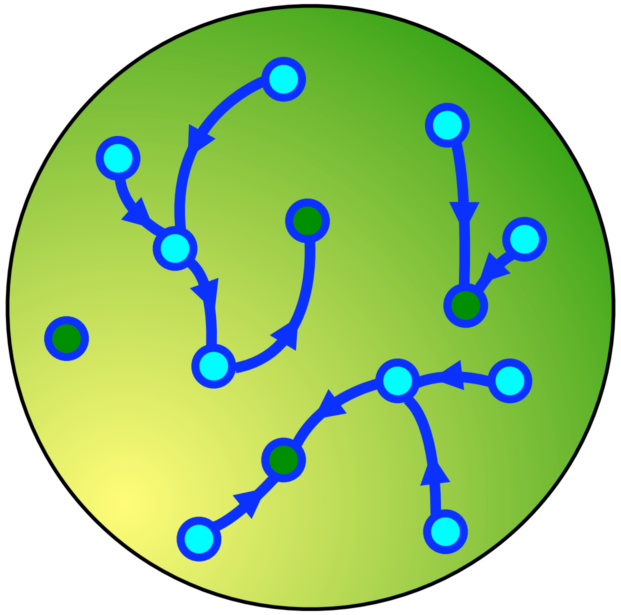
provides a worldsheet generalization of Hodges' matrix, but now valid for ***all N^k^MHV amplitudes***

- $$\Phi_{ij} = \frac{1}{x_{ij}} \left[\frac{\partial}{\partial Z_i}, \frac{\partial}{\partial Z_j} \right]$$ $\bar{\rho}\rho$ contractions

  $$\Phi_{ii} = -\sum_{j \neq i} \Phi_{ij} \prod_{a=0}^{d} \frac{y_a - x_j}{y_a - x_i}$$ YZ contractions

- $\{\omega_i(x)\}$ is a basis of the space of γ zero modes
- fixed vertex operators correspond to rows & columns absent from $\|\Phi\|^{r_1 \cdots r_{d+2}}_{c_1 \cdots c_{d+2}}$

What do these determinants actually mean?

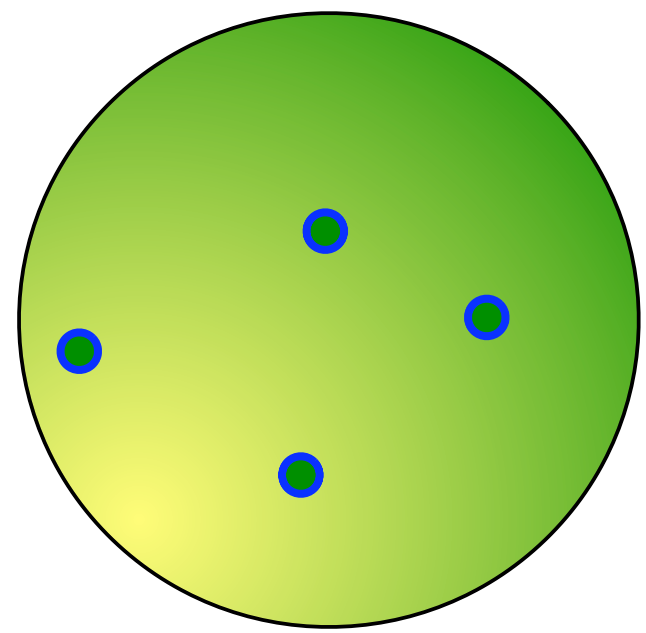


Rather than computing

$$\left\langle \prod_{k=1}^{d+2} \delta^2(\gamma) h_k \prod_{\ell=d+3}^{n} \left(\left[Y, \frac{\partial h_\ell}{\partial Z} \right] + \rho \frac{\partial}{\partial Z} \left[\bar{\rho}, \frac{\partial h_\ell}{\partial Z} \right] \right) \right\rangle$$

using the original free action, we can instead compute

$$\left\langle \prod_{k=1}^{d+2} \delta^2(\gamma) h_k(Z) \right\rangle$$



using the nonlinear action

$$S' = \int_\Sigma Y_I \left(\bar{\partial} Z^I + I^{IJ} \frac{\partial h}{\partial Z^J} \right) \; + \; \text{fermions}$$

obtained by exponentiating an integrated vertex operator

Path integral over Y imposes $\bar{\partial} Z^I + \{h\,,Z^I\} = 0$

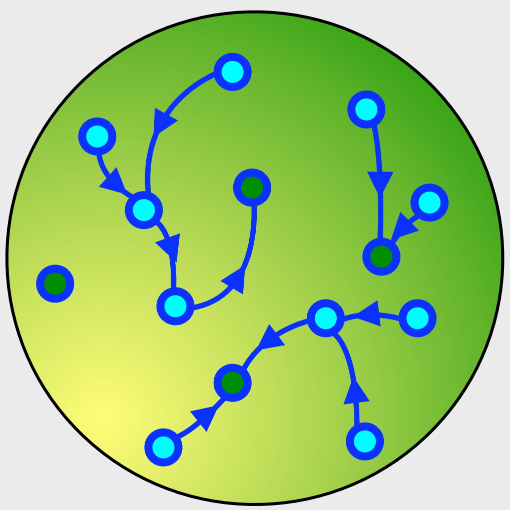
- perform field redefinition to $Z'(x)$, defined implicitly by

$$\bar{\partial} Z'^I(x) = \bar{\partial} Z^I(x) + \{h\,,Z^I(x)\}$$

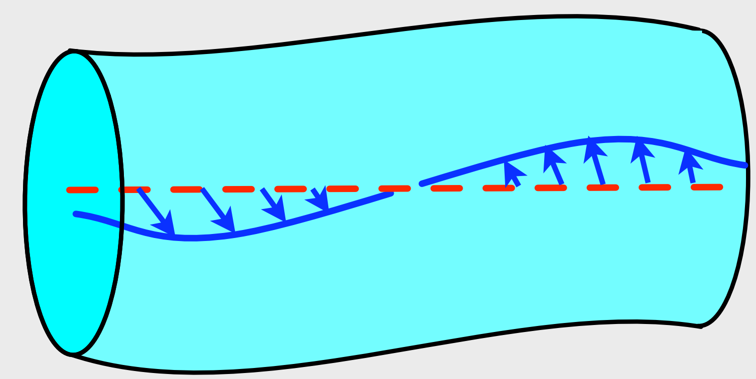
- Jacobian provided by fermion path integral *(c.f. Nicolai map)*

Expanding $h(Z(Z'))$ in fixed vertex ops[s] "grows a tree"

$$\sum$$



$$= \frac{\|\Phi\|^{r_1\cdots r_{d+2}}_{c_1\cdots c_{d+2}} \prod_{i=1}^{n} h_i}{\|\omega_j(x_{r_k})\|\;\|\omega_l(x_{c_m})\|} =$$



perturbative description of nonlinear graviton background

[Adamo,Mason; Casali,DS]

- Hodges determinant equivalent to sum over trees

  [Bern,Dixon,Perelstein,Rozkowsky; Nguyen,Spradlin,Volovich,Wen; Feng,He]

- form familiar from chiral bosonization

Combining all the ingredients, the $g = 0$ twistor string is just the statement that

all tree amplitudes in $\mathcal{N} = 8$ supergravity
are supported on degree d holomorphic maps

$$\mathcal{M}_{n,d} = \int \frac{\prod_{a=0}^{d} \mathrm{d}^{4|8} Z_a}{\mathrm{vol}(\mathrm{GL}(2))} \frac{\|\Phi\|^{r_1 \cdots r_{d+2}}_{c_1 \cdots c_{d+2}}}{\|\omega_j(x_{r_k})\| \, \|\omega_l(x_{c_m})\|} \mathrm{R}(\lambda_\alpha) \prod_{i=1}^{n} h_i(x_i)\, \mathrm{d}x_i$$

to curved twistor space **with infinity removed**

- precisely agrees with a representation of the classical gravitational S-matrix discovered last year[Cachazo,DS]

Conclusions

I have presented an holomorphic twistor string that computes the classical S-matrix of maximal supergravity

- anomaly free when $\mathcal{N} = 8$
- spectrum describes $\mathcal{N} = 8$ graviton supermultiplet
- integrated vertex operators give maps to nonlinear graviton

There are many open questions

- proper coupling to worldsheet gravity? other states?
- behaviour at higher genus?
- relation to $\mathcal{N} = 2$ superstring? [Berkovits,Ooguri,Siegel,Vafa]
- relation to "gravity = gauge $\times$ gauge"? [Bern,Carrasco,Johansson; *c.f.* Cachazo,Geyer]
- MHV diagrams from target effective field theory?
- other backgrounds (*e.g.* boundary correlators in AdS_4)?
- ...

Thank you