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Holographic Entanglement Entropy (HEE) of Excited States

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Based on

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arXiv:1212.4328 (JHEP 04(2013)051)

arXiv:1302.5703 (JHEP 05(2013)080)

arXiv:1304.7100

+ work in progress

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① Introduction

Use of Entanglement Entropy (EE)

- A quantum order parameter (~a generalization of \`Wilson loops’) ➡ Classify quantum phases.

  Entropy for non-equilibrium systems.

- A useful bridge between gravity and cond-mat. systems.

Gravity ⟺ Entanglement ⟺ Cond-mat. sysytems

$g_{\mu\nu}$ **AdS/CFT** $S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N}$ $|\Psi\rangle$

Holographic EE (HEE)

Information = Energy ?

1st law of thermodynamics: $T \cdot dS = dE$

Temp. Information Energy

⇒ Can we find an analogous relation in any quantum systems which are far from the equilibrium ?

Maybe: $T_{\mathrm{ent}} \cdot dS_A = dE_A$??

Information in A Energy in A

= EE

What ?

Can we observe EE ??

➡ The main motivation of this talk.

Holographic Entanglement Entropy (HEE) [Ryu-TT 06]

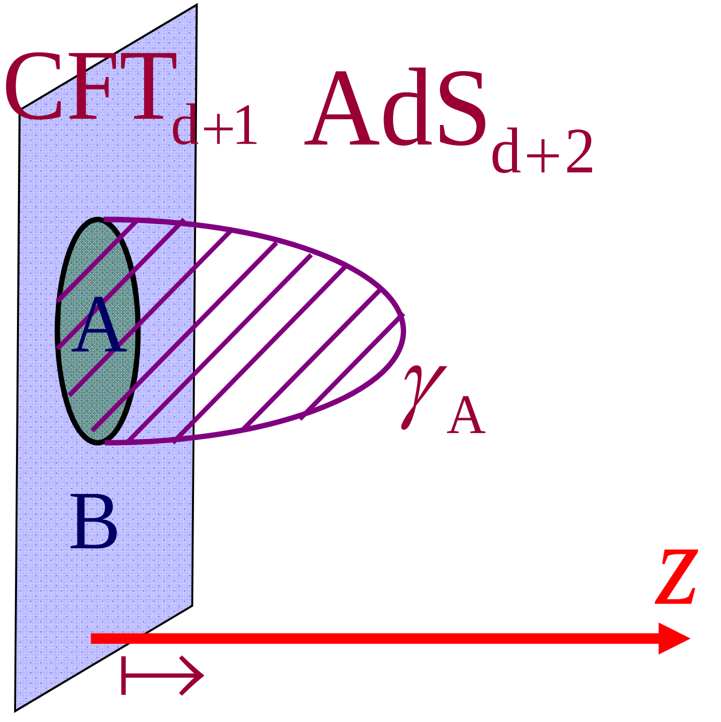
$$\boxed{S_A = \frac{\text{Area}(\gamma_A)}{4G_N}}$$

$$ds^2_{AdS} = R^2_{AdS}\frac{-dt^2 + \sum_{i=1}^{d} dx_i^2 + dz^2}{z^2}$$

γ_A is the minimal area surface (codim.=2) such that

$$\partial A = \partial \gamma_A \ \text{ and } \ A \sim \gamma_A \ .$$

homologous



$z > \varepsilon$ (UV cut off)

Note: In time-dependent b.g., we need to employ the covariant version [Hubeny-Rangamani-TT 07].

$\gamma_A \rightarrow$ extremal surface

Verification of HEE

- Confirmations of basic properties:

  Area law, Strong subadditivity (SSA), Conformal anomaly,….

- Direct Derivation of HEE from AdS/CFT:

  (i) Pure AdS, A = a round sphere [Casini-Huerta-Myers 11]

  (ii) Euclidean AdS/CFT [Lewkowycz-Maldacena 13, cf. Fursaev 06]

  (iii) Disjoint Subsystems [Headrick 10, Faulkner 13, Hartman 13]

  (iv) General time-dependent AdS/CFT → Not yet.

  [But, non-trivial evidences of SSA: Allais-Tonni 11, Callan-He-Headrick 12, Wall 13]

- Corrections to HEE beyond the supergravity limit:

  [Higher derivatives: Hung-Myers-Smolkin 11, de Boer-Kulaxizi-Parnachev 11,…..]

  [1/N effect: Barrella-Dong-Hartnoll-Martin 13]

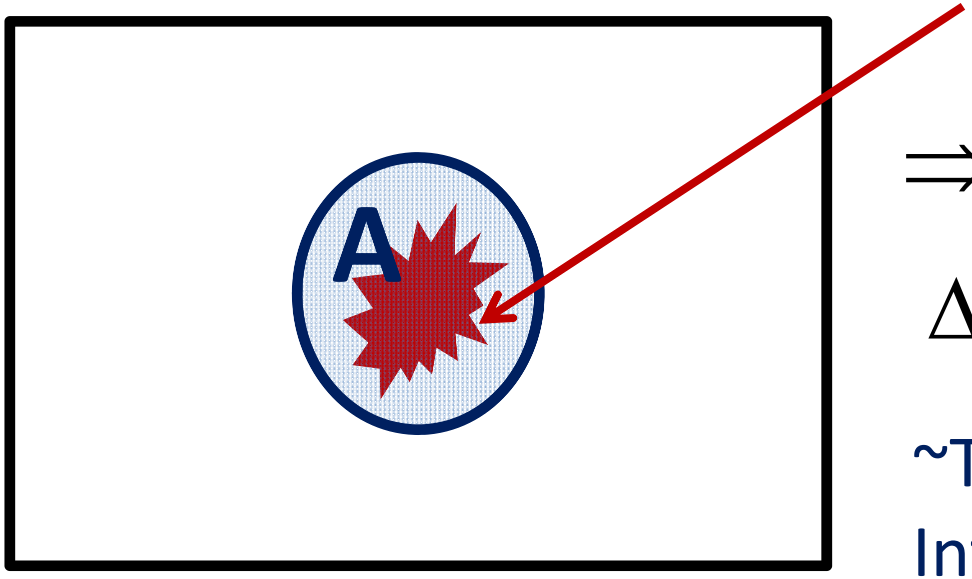
  [Higher spin gravity: de Boer-Jottar 13, Ammon-Castro-Iqbal 13]

Contents

② Local Excitations and HEE

[Nozaki-Numasawa-TT 13]

We would like to study **the relation between the EE and energy** when we excite the system **locally**.



Localized Excitations

$\Rightarrow$ Calculate

$$\Delta S_A = S_A^{Excited} - S_A^{Ground}$$

~The amount of quantum Information of excitations

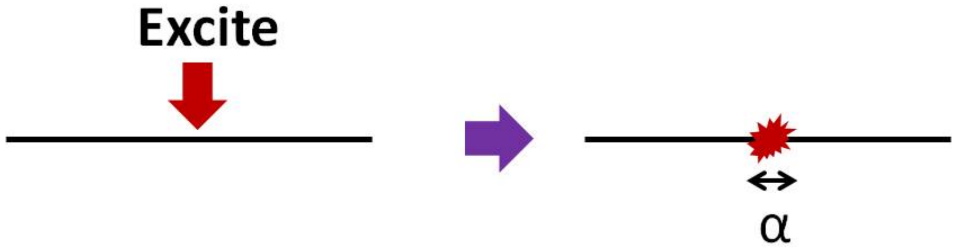
(UV finite)

Local Quenches (Local excitations)

Original Local Quench:

[Calabrese-Cardy 07]

Generalization
(Local Excitations):

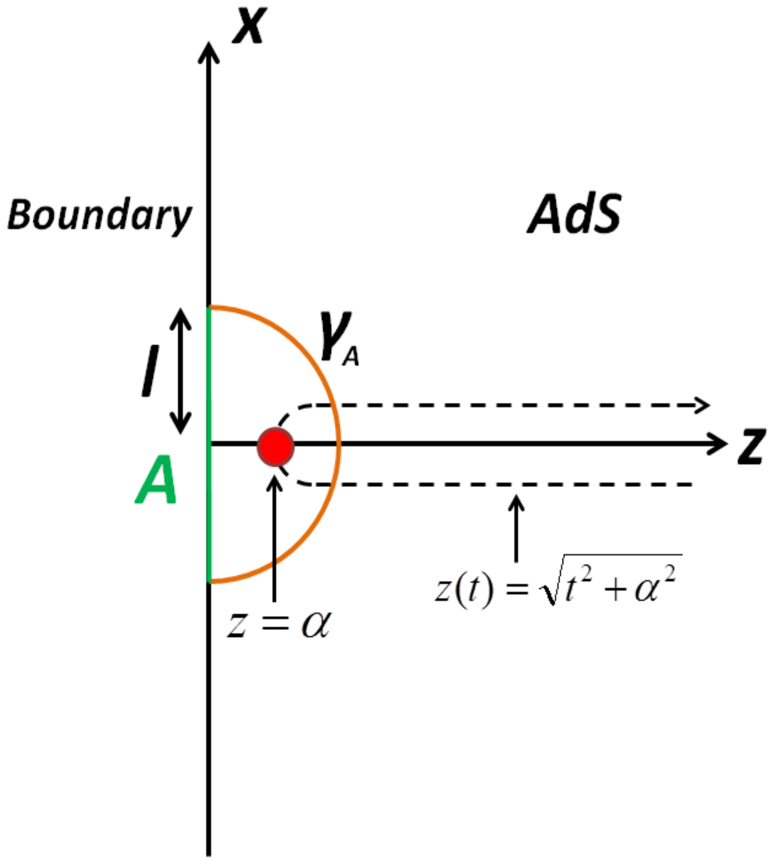


We consider this kind of example using the AdS/CFT.

A setup of holographic local quenches

A simple model of holographic local quench:

→ a free falling particle (mass **m**) in AdSd+2.



$$ds^2 = R^2 \cdot \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2}$$

Trajectory: $z(t) = \sqrt{t^2 + \alpha^2}$.

α~ the size of localized excitations

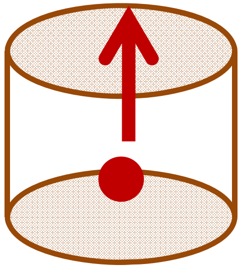
Construction of Back-reacted Solutions

We use the method noticed by [Horowitz-Itzhaki 99].

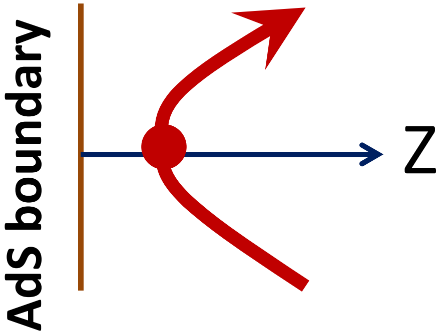
Start with the global AdS BH:

$$(M \propto G_N m R^2)$$

$$ds^2 = -f(r)d\tau^2 + \frac{R^2}{f(r)}dr^2 + r^2 d\Omega_d^2, \qquad f(r) = r^2 + R^2 - M/r^{d-2}.$$



Coordinate transformation



$$\sqrt{R^2 + r^2} \cdot \cos\tau = \frac{R^2 e^{\beta} + e^{-\beta}(z^2 + x^2 - t^2)}{2z},$$

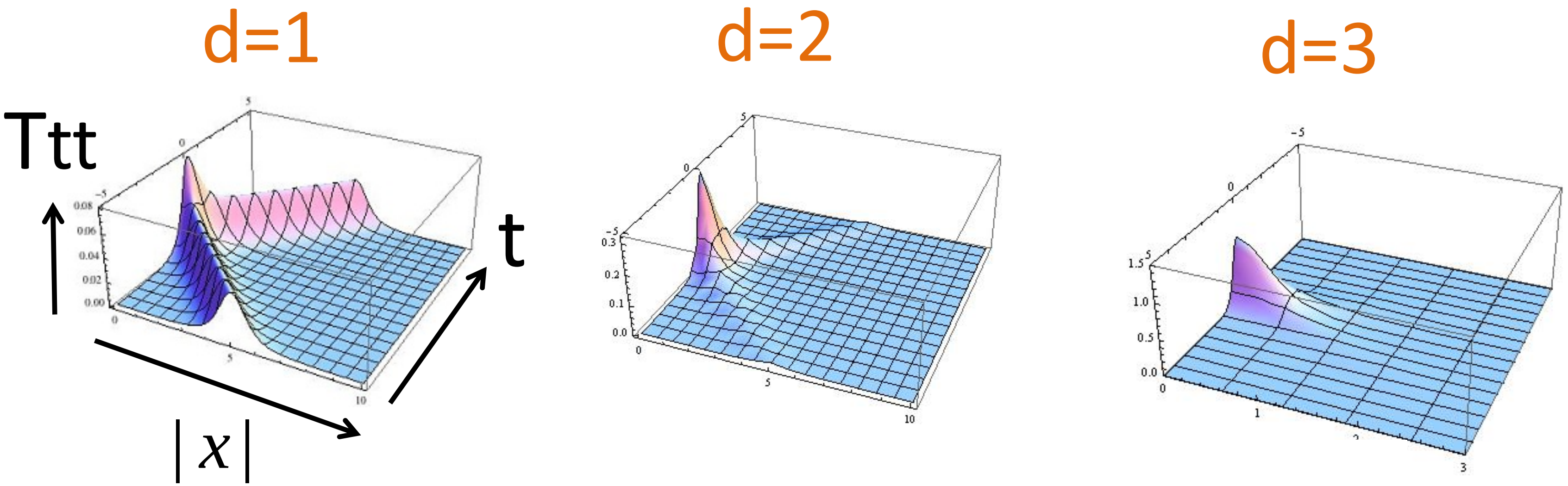
$$\sqrt{R^2 + r^2} \cdot \sin\tau = \frac{Rt}{z},$$

$$r\Omega_i = \frac{Rx_i}{z} \quad (i = 1,2,\ldots,d),$$

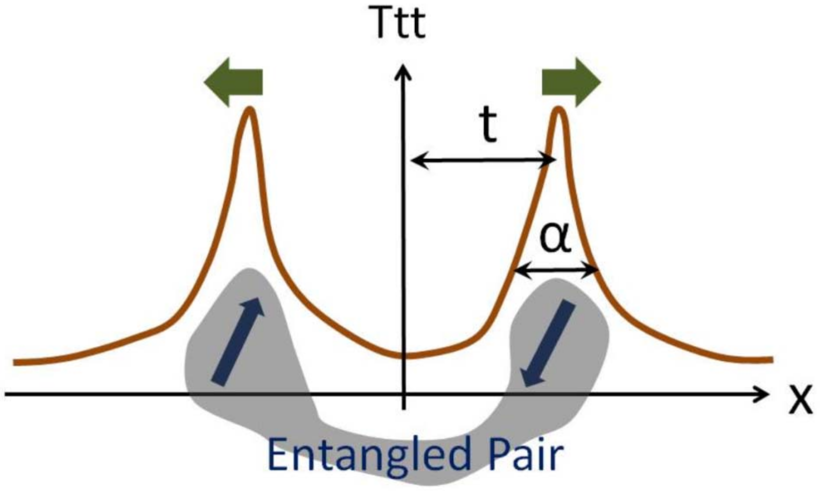
$$r\Omega_{d+1} = \frac{-R^2 e^{\beta} + e^{-\beta}(z^2 + x^2 - t^2)}{2z}.$$

Asymptotic AdS space (M=0 → Pure AdS)

Energy density via AdS/CFT



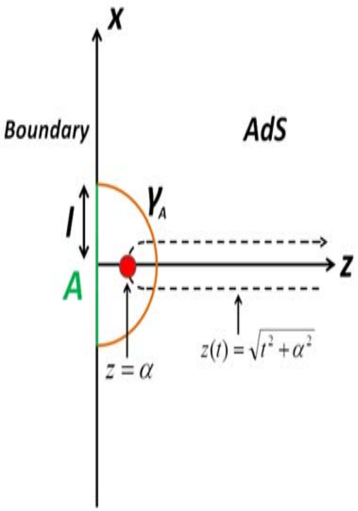
Basic picture:

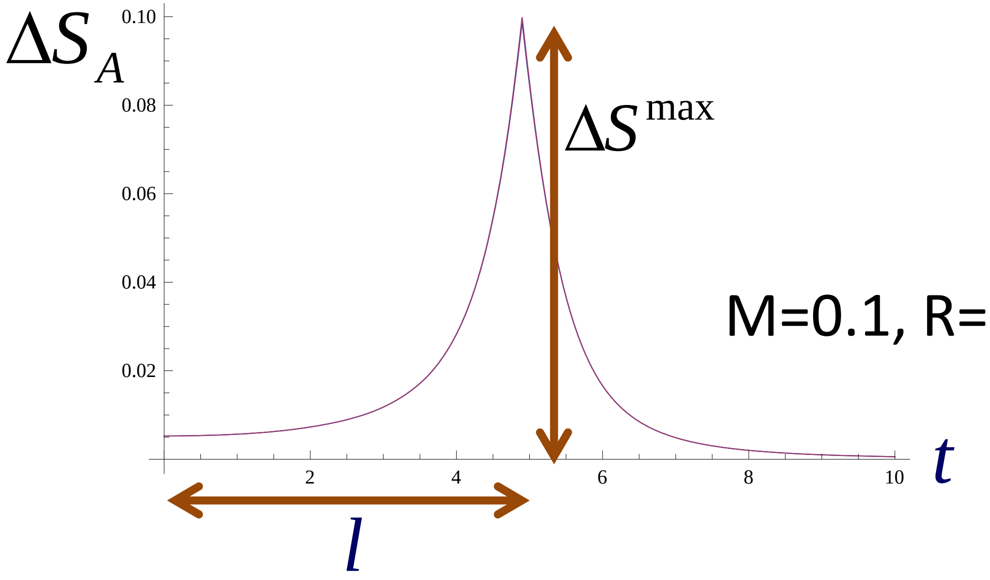


We can confirm this entanglement structure by calculating `entanglement density'.

Time Evolution of HEE in AdS3/CFT2

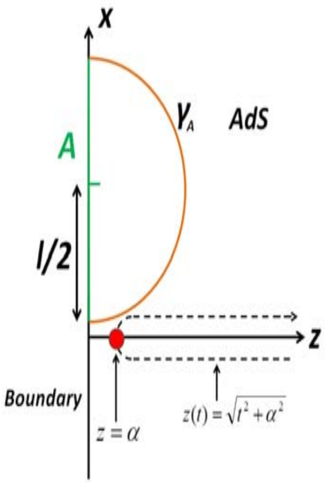
[Case 1]

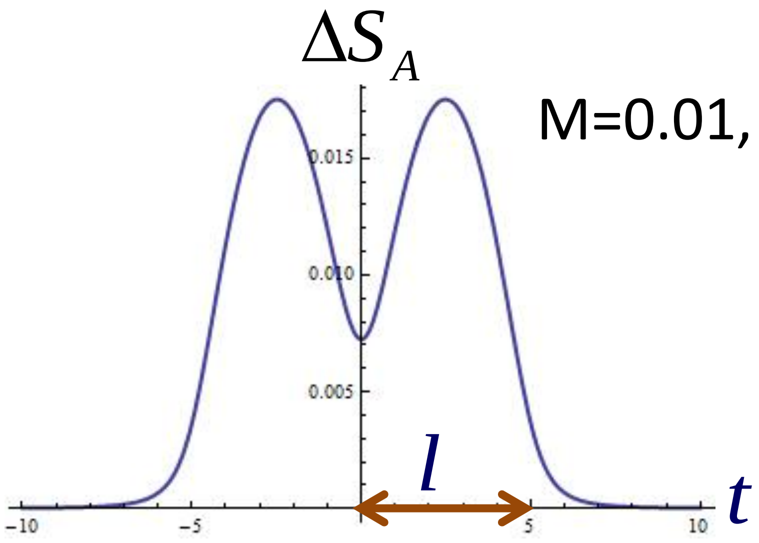




M=0.1, R=1, l=5

[Case 2]





M=0.01, R=1, l=5

Exact result in AdS3

Assume $M \ll R^2$

$$\Delta S^{\max} = \frac{c}{3}\log\left[\frac{R}{\sqrt{R^2-M}}\sin\left(\frac{\pi\sqrt{R^2-M}}{2R}\right)\right] \approx 2mR = 2\Delta.$$

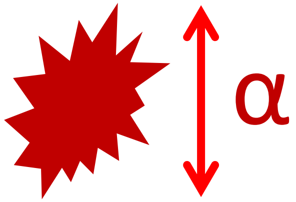
Summary:

The amount of quantum information

of Localized (weak) excitations in CFT ~ **E ▪ α**

Energy Size

(`fire ball' of gluons)

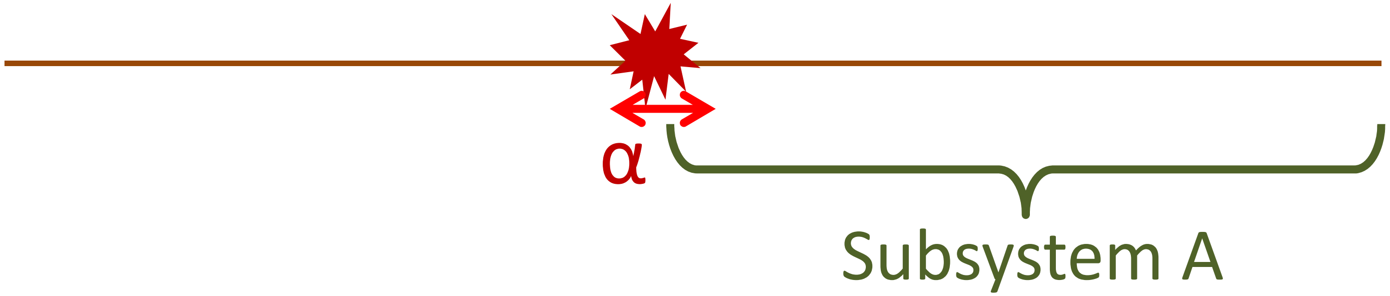


Actually, this relation is true in any dimension in AdS/CFT.

Comment

If we take the limit l>>t>>α in the case 2, we find

$$S_A = \frac{c}{6}\log\frac{t}{\alpha} + \frac{c}{3}\log\frac{l}{a} + \text{const.}$$



cf. 2d CFT result for local quenches by the joint procedure

$$S_A = \frac{c}{3}\log\frac{t}{a} + \frac{c}{6}\log\frac{l}{\alpha} + \text{const.}$$

③ Energy flow and HEE [Trivedi-Narayan-TT 12]

AdS plane waves: $ds^2_{(d+2)} = R^2 \cdot \frac{-2dx^+dx^- + Q\cdot z^{d+1}(dx^+)^2 + \sum_{i=1}^{d-1} dx_i^2 + dz^2}{z^2}.$

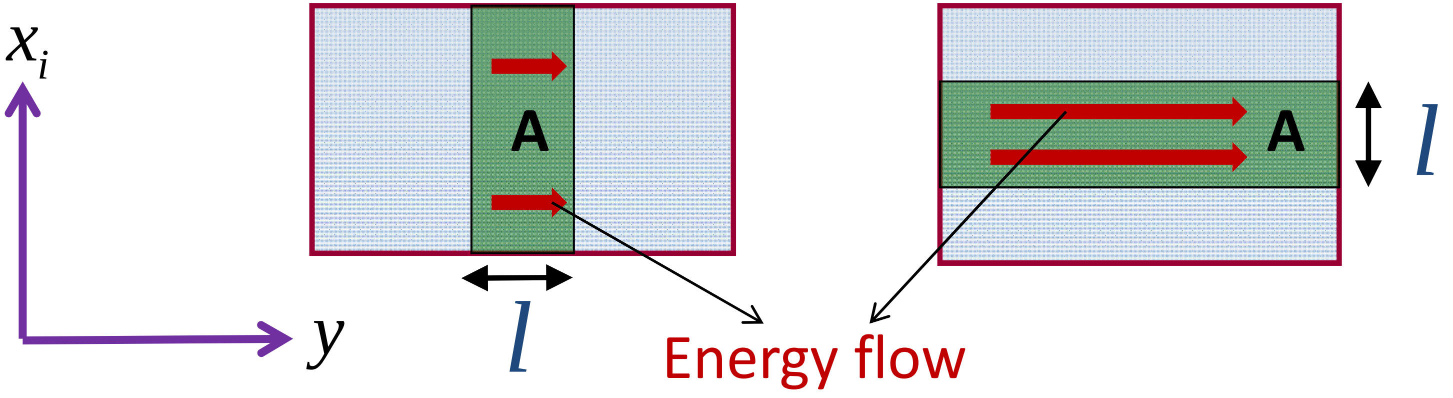
⇕ AdS/CFT

A simple model of excited states $T_{++} \propto Q > 0$ (energy flux).

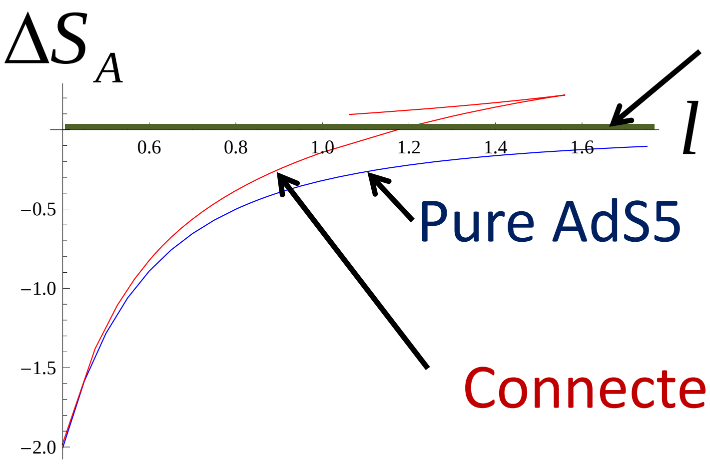
⇒ Two choices of strip subsystems in this anisotropic system

$(x^\pm = t \pm y)$

[Case 1] **[Case 2]**



[Case1] ⇒ There is a phase transition.



Disconnected solution in AdS wave

Pure AdS5

Connected solution in AdS5 wave

[Case2] ⇒ HEE increases due to the energy flux.

But smaller than thermal entropy. (`semi-extensive')

$$\Delta S_A \sim V_{d-1} \cdot \sqrt{Q} \cdot l^{(3-d)/2} \quad (< l^d).$$

In particular, d=3 (AdS5), we obtain $\Delta S_A \propto V_2 \cdot \log l$.

[⇒ 4 dim. Hyperscaling violating geometry with θ=1 via a null compactification, which was found in Narayan 12, Singh 12]

④ `1st Law' Relation between EE and Energy

[Bhattacharya-Nozaki-Ugajin-TT 12]

We want to find a universal relation in CFTs between

$$\Delta S_A = S_A^{Excited} - S_A^{Ground} \quad \text{and} \quad \Delta E_A = \int_A dx^d T_{tt}$$

in more general setups.

We will assume the excited state is (approximately) translationally invariant and isotropic.

Holographic Calculation

Consider an asymptotically AdS$_{d+2}$ background

(= an excited state in CFT$_{d+1}$):

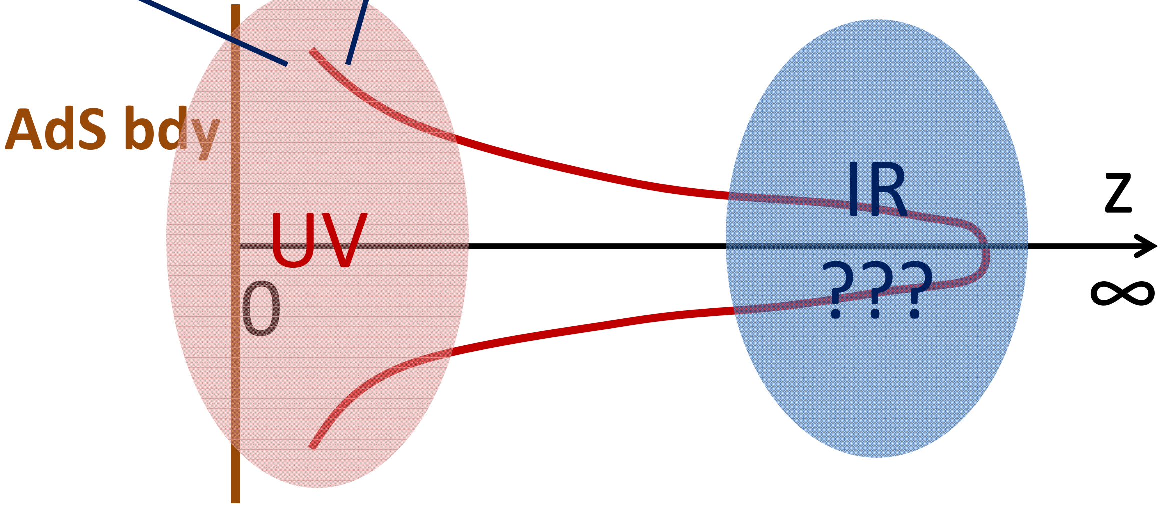
$$ds^2 = \frac{R^2}{z^2}\left(-f(z)dt^2 + g(z)dz^2 + \sum_{i=1}^{d} dx_i^2\right) \ ,$$

$$f(z) = 1 - mz^{d+1} + \ldots, \qquad g(z) = 1 + mz^{d+1} + \ldots..$$

$$\Rightarrow T_{tt} = \frac{dR^{d+1}m}{16\pi G_N}.$$

Energy density

We do not care the details of IR.



Holographic Prediction

Consider an exited state in a CFT which has an approximate translational and rotational invariance.

If the size of the subsystem A (= l) is small enough such that

$$T_{tt} \cdot l^{d+1} << R^{d} / G_{N} \approx O(N^{2}),$$

then the following `1st law' like relation is satisfied:

$$\underbrace{T_{ent} \cdot \Delta S_{A}}_{\textbf{Info.}} = \underbrace{\Delta E_{A}}_{\textbf{Energy}}, \qquad T_{ent} \equiv \frac{c}{l} \;,$$

Note 1: The constant c depends only on the geometry of A.

Note 2: For more general critical points with the dynamical exponent **z**, we have $T_{ent} = c \cdot l^{-z}$.

More Progresses

If the rotational invariance is broken, ΔS_A is a linear combination of not only Ttt but also other components of EM tensor.

[Pointed out in Guo-He-Tao 13, Allahbakhshi-Alishahiha-Naseh 13,
Blanco-Casini-Hung-Myers 13;
This problems does not occur when A= a round ball]

However, we can generally show the following equivalence

$$\Delta S_A = \Delta H_A \, (= -\mathrm{Tr}[\delta\rho_A \cdot \log \rho_{\mathrm{A}}]),$$

Modular Hamitonian

as pointed out in

[Blanco-Casini-Hung-Myers 13, Wong-Klich-Pando Zayas-Vaman 13]

⑤ What is the Einstein equation for HEE ?

[Nozaki-Numasawa-Prudenziati-TT 13, Bhattacharya-TT work in progress]

The `1st law-like' relation appears only when the size of A is small.

➡ What can we say if the size of subsystem is not small ?

This is related to a basic question in the AdS/CFT:

Gravity on AdS	↔	**CFT**
$g_{\mu\nu}(t,\vec{x},z)$	AdS/CFT	$S_A(t,\vec{x},l)$
Einstein eq.	↔	What ?

Below we study **a HEE counterpart of perturbative Einstein eq.** assuming small excitations of a CFT.

AdS4/CFT3

Let A be a round ball with radius l. Its center is situated at $(t,\vec{x})$.

The perturbative Einstein equation is rewritten as follows

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}=T_{\mu\nu}$$

Kinetic term

C.C.

Matter field contributions

$\phi\leftrightarrow O$

$$\boxed{\left(\partial_l^2-\partial_l-\partial_{\vec{x}}^2-\frac{3}{l^2}\right)\Delta S_A(t,\vec{x},l)=\langle O\rangle\langle O\rangle}$$

Note: There are **no time derivatives**.

It looks like an EOM for a scalar on a time slice of AdS4.

$\Rightarrow$ This gives a **constraint** for HEE at a fixed time.

The time evolution is determined by IR bdy conditions.

AdS4 Schwarzschild BH

When the size of the subsystem is very large , we find

$$\boxed{\left(\partial_l^2 - \partial_l - \partial_{\vec{x}}^2\right)\Delta S_A(t,\vec{x},l) = \langle O\rangle\langle O\rangle.}$$

This coincides with the holographic result for flat space.

AdS3/CFT2

In AdS3 gravity, we have two constraints:

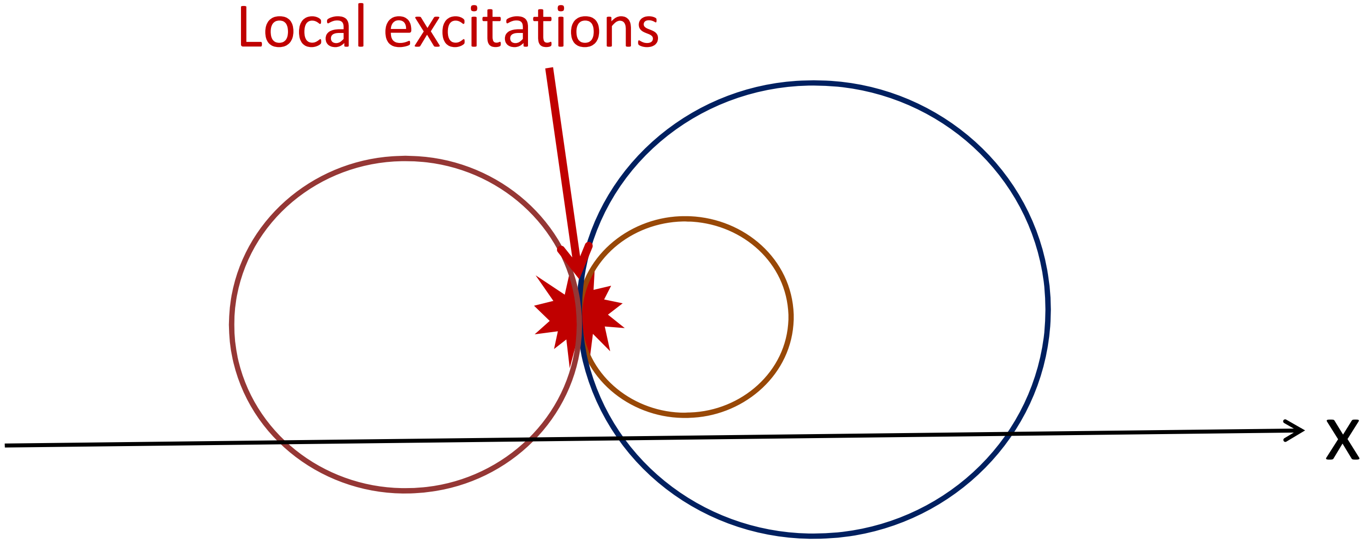
$$\boxed{\begin{aligned}&\left(\partial_x^2 - \partial_t^2\right)\Delta S_A(t,x,l) = \langle O\rangle\langle O\rangle,\\&\left(\partial_l^2 - \partial_t^2 - \frac{2}{l^2}\right)\Delta S_A(t,x,l) = \langle O\rangle\langle O\rangle.\end{aligned}}$$

[Confirmed in CFT2 by Wong-Klich-Pando Zayas-Vaman 13]

Intuitive interpretation of these constraints

Hyperbolic PDE: $\left(\partial_l^2 - \partial_{\vec{x}}^2\right)\Delta S_A(t,\vec{x},l) \approx 0$

$$\Rightarrow \Delta S_A \propto f(l-|x|)+g(l+|x|).$$



ΔS_A becomes non-trivial only when ∂A intersects with the excited region $\Leftrightarrow$ $l=\pm|x|$.

⑥ Conclusions

- We derived a universal relation between the EE and the energy of excited states for small subsystems in CFTs.

→ This looks analogous to the first law in thermodynamics.

What will happen in non-conformal theories ?

- For generic subsystems, the property of EE in a CFT depends on the details of the theory.

→ We found that the AdS/CFT relates the perturbative Einstein equation to a certain constraint equation of EE.

More precise interpretations of this constraint ?

Beyond the perturbation theory ?