Chern Simons Theories with fundamental matter and their bulk duals

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Introduction

- Pure Chern Simons theory is topological; has no local degrees of freedom. All degrees of freedom global. Interesting global dynamics but no local 3d physics Witten.
- Chern Simons minimally coupled to matter : no gluons but local matter degrees of freedom. CS fields mediate nonlocal interactions between propagating matter fields.
- Resulting theories dynamically rich. Fascinating non supersymmetric strong weak coupling dualities. Interesting bulk duals at large N. Novel high temperature saddle points (dual black holes ?). Also may implement anyonic statistics. Apparently have applications to the quantum hall effect. Worth studying.

Introduction

- This talk: study (generically non susy) large N $U(N)$ Chern Simons theories coupled to *fundamental* matter fields.
- N propagating degrees of freedom interacting with Chern Simons N^2 topological degrees of freedom. Turn out to be solvable in the t'Hooft large N limit at all values of the t' Hooft coupling $\lambda = \frac{N}{k}$.
- Nonetheless dynamically rich. This talk: nontrivial thermal phase structure. Evidence for strong weak coupling dualities between pairs of such theories.
- In the large N limit, fundamental theories closely related to $U(N)_k \times U(1)_{-k}$ bifundamental theories. Limit of $U(N)_k \times U(M)_{-k}$ theories with $\frac{M}{N} \to 0$. Limit of theories with bulk string duals (e.g. ABJ).
- Independent conjecture for bulk duals of fundamental theories in terms of parity violating Vasiliev equations. Implies Vasiliev= limit of string theory.

The Theories

In the first part of this talk we study the most general 'renormalizable' $U(N)$ Chern Simons theory interacting with a single fundamental boson and single fundamental fermion.

$$S = \int d^3x \Big[i\varepsilon^{\mu\nu\rho} \frac{k}{4\pi} \mathrm{Tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) + D_\mu \bar{\phi} D^\mu \phi + \bar{\psi}\gamma^\mu D_\mu \psi + m_B^2 \bar{\phi}\phi + m_F \bar{\psi}\psi + \frac{4\pi b_4}{k}(\bar{\phi}\phi)^2 + \frac{4\pi^2 x_6}{k^2}(\bar{\phi}\phi)^3 + \frac{4\pi x_4}{k}(\bar{\psi}\psi)(\bar{\phi}\phi) + \frac{2\pi y_4'}{k}(\bar{\psi}\phi)(\bar{\phi}\psi) + \frac{2\pi y_4''}{k}\left((\bar{\psi}\phi)(\bar{\psi}\phi) + (\bar{\phi}\psi)(\bar{\phi}\psi)\right) \Big].$$

Parameters: m_B, m_F, b_4 (massive) x_6,x_4 y_4', y_4'' (dimensionless), + N, k (discrete). In the t' Hooft limit new effective continuous parameter $\lambda = \frac{N}{k}$.

Theories: N=2 point

Useful to focus on special cases and limits. First special case

$$m_F = m_B = b_4 = y_4'' = 0 \quad x_4 = x_6 = y_4' = 1 \tag{1}$$

$$S = \int d^3x \Big[i\varepsilon^{\mu\nu\rho} \frac{k}{4\pi} \mathrm{Tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) + \widetilde{D}_\mu \bar{\phi} \widetilde{D}^\mu \phi + \bar{\psi} \gamma^\mu \widetilde{D}_\mu \psi$$
$$+ \frac{4\pi}{k} (\bar{\psi}\psi)(\bar{\phi}\phi) + \frac{2\pi}{k} (\bar{\psi}\phi)(\bar{\phi}\psi) + \frac{4\pi^2}{k^2} (\bar{\phi}\phi)^3 \Big]$$

$\mathcal{N} = 2$ susy CS interacting with a single fundamental chiral multiplet; no superpotential. Superconformal, well studied. Gaiotto, Yin.

Theories: Fermionic and Bosonic limits

- Other special limits obtained by decoupling the boson or fermion. Sending the boson mass to infinity yields

$$S = \int d^3x \left[i\varepsilon^{\mu\nu\rho} \frac{k}{4\pi} \mathrm{Tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) + \bar{\psi}\gamma^\mu \widetilde{D}_\mu \psi + m_F^{\text{reg}} \bar{\psi}\psi \right]. \tag{2}$$

  Theory conformal when $m_F^{\text{reg}} = 0$

- Sending the fermionic mass to infinity yields

$$S = \int d^3x \left[i\varepsilon^{\mu\nu\rho} \frac{k}{4\pi} \mathrm{Tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) + \widetilde{D}_\mu \bar{\phi} \widetilde{D}^\mu \phi + \sigma(\bar{\phi}\phi + \frac{m_B^{\text{cri}}}{4\pi}) \right] \tag{3}$$

  Chern Simons gauged massive critically coupled bosons Jain, S.M. Yokoyama. Conformal when $m^{\text{cri}} = 0$.

Thermal Partition Function

- Full class of theories above likely 'solvable'. In particular the finite temperature partition function on S^2 has been computed for most general theory. Will now describe how.
- Wish to compute path integral on $S^2 \times S^1$. We organize the computation as follows. First compute the effective action as a function of the holonomy and commuting 2d gauge fields. Next integrate over these fields.
- Effective action sum over bubble graphs; may be graded in powers of N (roughly by number of fundamental index loops). Graphs with no index loops - pure Chern Simons theory - contribute at $\mathcal{O}(N^2)$. Graphs at $\mathcal{O}(N)$ (roughly those with one fundamental index loop). Graphs at $\mathcal{O}(N^0)$ or lower; negligible in the large N limit.

Partition Function : S_{eff}

- The contribution to the effective action of graphs at order N (roughly one fundamental loop) takes the schematic form

$$S_{eff} = NT^2 \int d^2x \left(f_1(U) + R\frac{f_2(U)}{T^2} + \frac{f_3(U)}{T^2}(\partial U)^2 \right. \\ \left. \frac{f_4(U)}{T^4} f_{ij} f^{ij} + \ldots \right) \tag{4}$$

Aharony, Giombi, Gur-Ari, Maldacena, Yacoby

- S_{eff} is of the same order as the contribution from pure Chern Simons theory only when $VT^2 = \mathcal{O}(N)$. For this reason we keep $\frac{T^2 V}{N}$ fixed in the limit $N \to \infty$.

- With this scaling

$$S_{eff}(U(x)) = NT^2 \int d^2x \, f_1(U) + \text{subleading}.$$

To compute S_{eff} it is thus sufficient to set $f_{ij} = 0$, $U(x) = U$ and work in flat space.

- To complete the determination of the partiton function we must now add in the contribution of all graphs with no fermion loops and then integrate over all U and f_{ij}. These two steps together, however, simply amount to computing

$$Z = \langle e^{-S_{eff}[U]} \rangle_{N,k}$$

where the expectation value $\langle\rangle$ is taken in *pure* Chern Simons theory of rank N and level k. Jain, S.M., Sharma, Takimi, Wadia Yokoyama

Partition Function: Level Rank duality

- Recall that $S_{eff}(U)$ is a gauge invariant function of the holonomy. Every such function can be expanded as a linear sum of Wilson loops (in arbitrary representations of $U(N)$) around the time circle. It follows from the well known transformation rules for Wilson loops under level rank duality Naculich, Schnitzer of pure Chern Simons theory that

$$Z = \langle e^{-S_{eff}[\mathrm{Tr}U^n]} \rangle_{N,k} = \langle e^{-S_{eff}[(-1)^{n+1}\mathrm{Tr}\tilde{U}^n]} \rangle_{|k|-N,-k}$$

- In the large N limit $S_{eff} = Ns[\rho]$ where $\rho(\alpha)$ is the eigenvalue density function. Level rank duality implies that

$$Z = \langle e^{-Ns[\rho]} \rangle_{N,\lambda} = \langle e^{-\tilde{N}s[\tilde{\rho}]} \rangle_{\tilde{N},\tilde{\lambda}}$$

where

$$\tilde{N} = |k| - N, \quad \tilde{\lambda} = \lambda - \mathrm{sgn}(\lambda), \quad \tilde{\lambda}\tilde{\rho}(\alpha) + \lambda\rho(\alpha + \pi) = \frac{1}{2\pi}$$

Jain, S.M., Sharma, Takimi, Wadia Yokoyama

Partition Function: Matrix model

- Returning to the evaluation of the partition function, by adapting a beautiful older paper of Blau and Thompson it may be shown that, in the large N limit,

$$Z = \langle e^{-S_{eff}[U]} \rangle_{N,k} = \int dU e^{-S_{eff}[U]}$$

where the matrix integral in the last line is evaluated by saddle points, subject to the restriction that everywhere

$$\rho(\alpha) \leq \frac{1}{2\pi|\lambda|}$$

Aharony, Giombi, Gur-Ari, Maldacena, Yacoby; Jain, S.M., Sharma, Wadia, Takimi, Yokoyama

- The inequality is an immediate consequence of the duality transformation formula for ρ coupled with the obvious inequalities $\rho \geq 0, \quad \tilde{\rho} \geq 0$. New kind of matrix model; solvable in the large N limit Jain, S.M., Sharma, Wadia Yokoyama; Takimi.

- As explained, $S_{eff}(U)$ refers to the sum of all bubble graphs at $\mathcal{O}(N)$ (roughly one fundamental index loop). Quite remarkably, it turns out to be possible to analytically sum these graphs, in an unusual 'lightcone' gauge, in the most general fundamental Chern Simons matter theory. Giombi, S.M., Prakash, Trivedi, Wadia, Yin
- The result of this summation is given in terms of a 'free energy functional'

$$F_{eff}(c_S, c_F, \rho)$$

Jain, Trivedi, Wadia, Yokoyama; Giombi, S.M., Prakash, Jain, S.M., Sharma, Wadia; Takimi, Yokoyama, Jain, S.M., Yokoyama where c_B and c_F respectively are the thermal masses of the bosonic and fermionic fundamental fields. S_{eff} is determined by extremizing this free energy functional w.r.t the thermal masses c_F, c_B. The equations for thermal masses that follow from the extremization of the free energy functional are called gap equations.

E.g.: Pure Fermi Theory

$$F[\rho, c_F] =$$

$$\frac{1}{6\pi}\left[|c_F|^3\frac{(\lambda - \text{sgn}(\lambda))}{\lambda} + \frac{3}{2\lambda}c_F^2\hat{m}_F^{\text{reg}} - \frac{1}{2\lambda}\frac{(\hat{m}_F^{\text{reg}})^3}{(\lambda - \text{sgn}(m_F^{\text{reg}}))^2}\right.$$

$$\left. - 3\int_{-\pi}^{\pi} d\alpha\rho(\alpha)\int_{|c_F|}^{\infty} dy\, y(\ln(1+e^{-y-i\alpha-\nu_F}) + \ln(1+e^{-y+i\alpha+\nu_F}))\right].$$

Gap Equation

$$\mathcal{C} = \frac{\text{sgn}(\lambda)\ |c_F| - \hat{m}_F^{\text{reg}}}{2\lambda}$$

$$\mathcal{C} = \frac{1}{2}\int d\alpha\rho(\alpha)$$

$$\left(\log(2\cosh\frac{|c_F| + i\alpha + \nu_F}{2}) + \log(2\cosh\frac{|c_F| - i\alpha - \nu_F}{2})\right)$$

Giombi, S.M., Prakash, Trivedi, Wadia, Yin; Aharony, Giombi, Gur-Ari, Maldacena, Yacoby; Jain, S.M., Sharma, Wadia, Takimi, Yokoyama, Jain, S.M. , Yokoyama.

$F_{eff}[c, \rho]$ has been computed for the most general renormalizable theory with one fundamental scalar and fermion Jain, S.M. , Yokoyama It turns out that

$$F_{eff}(c_F, c_B, \rho) = F_{\tilde{\rho}}(\tilde{c}_F, \tilde{c}_B, \tilde{\rho})$$

$$\tilde{c}_F = c_B, \quad \tilde{c}_B = c_F$$

$$\tilde{N} = |k| - N, \quad \tilde{k} = -k \quad \tilde{x}_4 = \frac{1}{x_4}, \quad \tilde{m}_F = -\frac{m_F}{x_4}$$

$$\tilde{x}_6 = 1 + \frac{1 - x_6}{x_4^3}, \quad \tilde{b}_4 = -\frac{1}{x_4^2}(b_4 + \frac{3}{4}\frac{1 - x_6}{x_4} m_F)$$

$$\tilde{m}_B^2 = -\frac{1}{x_4} m_B^2 + \frac{3}{4}\frac{1 - x_6}{x_4^3} m_F^2 + \frac{2}{x_4^2} b_4 m_F$$

It follow that theories related by this parameter map have identical partition functions. Suggests duality.

Immediate consistency checks

- Reduces to the Giveon Kutasov level rank duality for the $\mathcal{N} = 2$ superconformal theory.
- Proposed duality relations map the $\mathcal{N} = 1$ manifold of theories

$$m_F = \mu, m_B^2 = \mu^2, b_4 = \mu w, x_4 = \frac{1+w}{2}, \\ x_6 = w^2, y_4' = w, y_4'' = w - 1 \tag{5}$$

onto themselves with Jain, S.M. , Yokoyama

$$w' = \frac{3-w}{1+w}, \quad \mu' = -\frac{2\mu}{1+w}.$$

- Historically duality was first conjectured in special cases not from thermal partition functions but from matching of spectra and correlators of single sum operators at special at conformal points, see below.

Decoupling limits

- The procedure for the computation of the thermal free energy also yields the thermal masses, c_F and c_B. At $T = 0$ these are the exact pole masses the propagating fields.
- It is possible to scale the 3 dimensionful parameters in our general Lagrangian so that, for instance, the bosonic pole mass stays fixed at $m^{\rm cri}$ while the fermionic pole mass is sent to infinity. In this limit the free energy functional of our system turns out to be indpendent of the 'spurious' parameters x_6, x_4, and agrees precisely with the free energy functional of the critical boson theory.
- Very similar results apply to the 'fermionic' limit under which the fermionic pole mass is kept fixed while the bosonic pole mass is scaled to infinity.

Duality of fermionic and bosonic scaling limits

- It turns out that the fermionic and bosonic decoupling limits are interchanged by our duality map Jain, S.M. , Yokoyama. This implies a duality between the purely fermionic and purely bosonic theories with N and k interchanged according to the level rank relations, while

$$m_B^{\text{cri}} = -\frac{1}{\lambda - \text{sgn}(\lambda)} m_F^{\text{reg}} \qquad (6)$$

cri

Aharony, Giombi, Gur-Ari, Maldacena, Yacoby

- The bosonization duality is thus an extreme limit of a massive deformation of Giveon Kutasov duality Jain, S.M. , Yokoyama. This observation strongly suggests that the bosonization duality above is also true at large but finite N, and likely also for bifundamental theories at small $\frac{M}{N}$.
- This bosonization duality was first proposed, in the massless limit, from the study of the spectrum and correlators of single sum operator of these theories as we now explain.

Single sum spectrum

- In the free limit the single sum spectrum of both the conformal fermionic and the conformal (critical) bosonic theory is given by

$$(2,0)+\sum_{j=1}^{\infty}(j+1,j)$$

where we label primary operators under the conformal group by $(\mathrm{dimension}, \mathrm{spin})$.

- $(j+1,j)$ are all short representations of the conformal algebra, corresponding to conserved higher spin currents. This fact may be used to demonstrate that the scaling dimensions of these operators are not renormalized at leading order in $\frac{1}{N}$ Giombi, S.M., Prakash, Trivedi, Wadia, Yin; Aharony, Gur-Ari, Yacoby. In other words the spectrum of single sum spectrum operators is identical in purely fermionic and purely bosonic theories at all values of the t' Hooft coupling.

3 point functions of single sum operators

- This non renormalized spectrum, together with the some perturbative results on three point functions, together with the nature of the bulk duals, motivated the first suggestion of a fermi-bose level rank dualities between the purely fermionic and bosonic theories Giombi, S.M., Prakash, Trivedi, Wadia, Yin; Aharony, Gur- Ari, Yacoby.
- The initially speculative suggestion was given a much firmer basis in a remarkable couple of papers by Maldacena and Zhiboedov. These authors used the approximately conserved higher spin currents present in both these theories, to almost completely determine the form of their three point functions. In more detail they demonstrated that these three point functions, in both theories, is determined upto a 2 parameter ambiguity

$$< j_1, j_2, j_3 >= f(j_1, j_2, j_3, \tilde{\lambda}, \tilde{N}) >$$

3 point functions of single sum operators

- It follows that the three point functions of the bosonic theory takes the above form with $\tilde{\lambda}$ and $\tilde{N}$ given by some function of λ_F and N_F. Similarly the three point functions of the bosonic theory takes the same form with $\tilde{\lambda}$ and $\tilde{N}$ given by distinct functions of λ_B and N_B. Equating yields a mapping between λ_F, N_F and λ_B, N_B under which three point functions are preserved.
- By explicitly determining these correlators in special kinematical limits in each theory, Aharony, Gur-Ari and Yakobi demonstrated that this map is indeed that of level rank duality

$$N_B = |k_F| - N_F, \quad k_B = -k_F$$

- The ageement of both correlators as well as thermal free energies at conformal points Aharony, Giombi, Gur-Ari, Maldacena, Yacoby together constitute compelling evidence for this duality.

Phases of the finite temperature partition function

- Let us return to the computation of the thermal partition function. With $S_{eff}(U)$ in hand we need to evaluate the capped matrix integral

$$\int dU e^{-S_{eff}[U]}$$

in the large N limit

- A general set of techniques to solve such matrix models has been developed Jain, S.M., Sharma, Wadia, Takimi, Yokoyama. and explicitly implemented in the case of conformal fermionic, bosonic and supersymmetric theories Takimi.
- While the details are involved, the the main qualitative results are easy to understand; I explain them with the aid of a few diagrams.

Phases

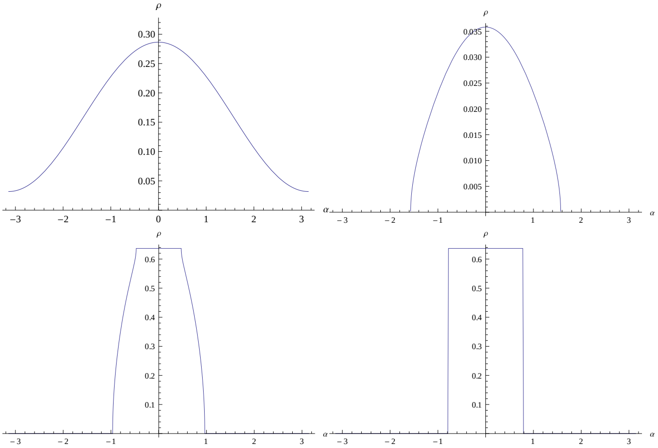

Figure: The eigenvalue distribution of the capped Gross Witten Wadia (GWW) model at $\lambda = 0.25$ for increasing values of ζ:For $\zeta = \frac{1}{2},\ 2,\ 4.6,\ 11.03$ respectively.

Phases

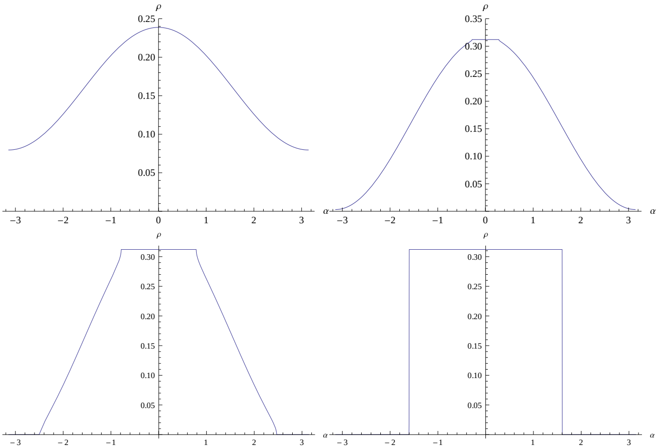

Figure: The eigenvalue distribution of the capped GWW model at (for $\lambda > \lambda_c$) $\lambda = 0.51$ for increasing values of ζ:For $\zeta = \frac{1}{2}, 0.98, 1.13, 5.06$ respectively.

Bulk dual

- In the free limit, the conformal Chern Simons theories we have studied above all have bulk duals in terms of Vasiliev's equations of higher spin dynamics. Klebanov, Polyakov; Sezgin, Sundell; Giombi, Yin.
- It has been conjectured that the effect of boundary interaction λ is of to change the 'Vasiliev interaction phase;' by $\frac{\pi\lambda}{2}$. Giombi, S.M., Prakash, Trivedi, Wadia, Yin;
- Nontrivial evidence for this conjecture from the successful construction of Vasiliev duals of susy theories. Works even for the $\mathcal{N} = 6$ theory. Dual of the $U(N) \times U(M)$ ABJ theory involves a Vasiliev theory whose fields are all $M \times M$ matrices. Chang, S.M. , Sharma, Yin
- However ABJ theory is also dual to string theory. How can a single theory have two bulk duals?

Vasiliev's fields as string bits

- According to usual AdS/CFT lore, the dual of a string state is an operator schematically denoted by $TrABABAB$ where A is any bifundamental field while B is the reverse. What is the Vasiliev description of this string?
- Note that the bulk t' Hooft coupling of the bulk Vasiliev equations is $\propto \frac{M}{N}$. The classical Vasiliev description applies only when $M \ll N$. In this limit boundary $U(M)$ interactions are weak compared to $U(N)$ interactions. Can be ignored for many purposes except for Gauss law.
- Consequently $TrABABAB$ effectively 'breaks up' into a collection of weakly interacting 'single sum' operators AB. There is a one to one correspondence between the $U(M)$ adjoint operators AB and Vasiliev's fields.
- We conclude that when $M \ll N$, the fundamental string breaks up into bits. Vasiliev's equations describe the dynamics of these bits, the partonic constituents of the fundamental string.

Fundamental Strings as Vasiliev Flux tubes.

- Viewed from the reverse direction, interaction effects in Vasiliev theory begin to sew up the different Vasiliev fields into necklaces that yield the fundamental string.
- Perhaps the following words apply. As $\frac{M}{N}$ increases, kinematical confinement (from the Gauss Law) turns into dynamical confinement. The gauge flux tube develops a tension much larger than the inverse AdS radius. This flux tube is the fundamental string of string thoery. Strings emerge from the confining dynamics of the non abelian Vasiliev theory.
- Very interesting if true, as the fundamental string has no internal structure (thickness) unlike a usual QCD string.

Thermodynamics of free ABJ on S^2

- Claim that glue between B and A partons much weaker than A and B when $M \ll N$ can be made quantitatively precise in the computation of the S^2 parition function in the *free* ABJ theory in the large N limit with $\frac{M}{N}$ taken to be an arbitrary parameter. Because we work with the free theory, no analogue of the bound on eigenvalue density.
- Even in this limit we must account for interactions between matter and the Polyakov line of $U(N)$ and $U(M)$ gauge fields (i.e. the Gauss law). We find that the theory undergoes *two* phase transitions as a function of temperature.
- Low temperature: confined phase, gase of single traces $\mathrm{Tr}(A_1 B_1 A_2 B_2 \ldots A_m B_m)$ or closed strings.
- Raising temperature: first order phase transition at a temperature of order unity. Above phase transition temperature, $U(M)$ deconfines but $U(N)$ continues to confine.

Strings fall apart

- Intermediate temperature phase exactly described by $U(M)$ with matter content $A_i B_j$ in the deconfined phase. Matches spectrum of Vasiliev's fields.
- In other words: traces (single strings) of the low temperature phase disintegrate into a free gas of single sum operators (Vasiliev's particles).
- Further raising the temperature: another phase transition (GWW, 3rd order). Transition temperature of order $\sqrt{\frac{N}{M}}$. Associated with the 'deconfinement' of the gauge group $U(N)$.
- At temperatures much higher than this transition temperature, system approximately plasma of A_i and B_j. I.e. the single sums $\mathrm{Tr}(A_i B_j)$, of the intermediate temperature phase, break up into their basic building blocks in the high temperature phase. Presumably dual to a black hole in the bulk theory.

Conclusions

- Matter Chern Simons theories are very rich systems. Analytically tractable in the large N limit when matter is in the fundamental representation. Strong evidence of level rank dualities in this limit. All dualities may be regarded as massive deformations of the susy Giveon Kutasov duality. Explicit results for thermal partition function as function of coupling. Qualitatively new phases and physics.
- Generalization to bifundamental theories. Weakly coupled dual description in terms of non abelian Vasiliev equations at small $\frac{M}{N}$, and string theory for $\frac{M}{N}$ of order unity. String - Vasiliev duality. Vasiliev fields as string bits.

Outlook

This investigation has just begun. In the future it would be nice to have

- Computations in another gauge, e.g. temporal gauge.
- Better understanding (proof?) of level rank duality.
- Generalization of finite temperature computations to bifundamental matter atleast at small $\frac{M}{N}$.
- Computation of exact S matrix of massive theories.
- Thorough investigation of the physics of bosonization (e.g. Fermi Sea= Bose Condensate? Anyonic statistics?).
- Resolution of puzzles with Vasiliev duality (why is the phase not a function).
- More examples of nonsusy dualities obtained from mass deformations of conformal susy dualities (e.g. $N = 4$ YM?)
- Dual of bifundamental fermions at $M = N$?

Sounds like an exciting programme.