Max Planck Institute for Mathematics
California Institute of Technology

# 4d-2d correspondence

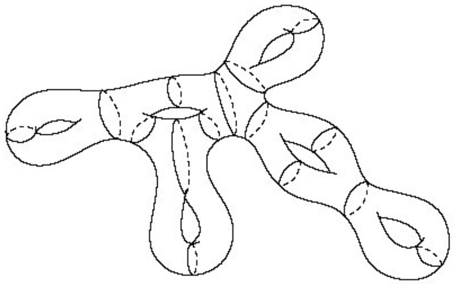
Sergei Gukov

based on: arXiv:1302.0015 ("bottom-up approach")
with A.Gadde and P.Putrov + "top-down approach"

- **Class S:**

2-manifold $C$ ➡ 4d $\mathcal{N}=2$ theory

**T[C]**

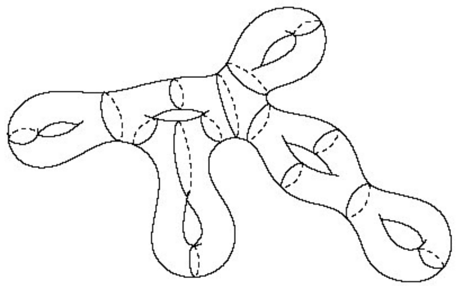


[D.Gaiotto, G.Moore, A.Neitzke]
[D.Gaiotto]
[L.F.Alday, D.Gaiotto, Y.Tachikawa]

- **Class S:**

2-manifold $C$ → 4d $\mathcal{N}=2$ theory

$T[C]$



[D.Gaiotto, G.Moore, A.Neitzke]
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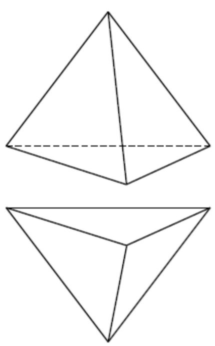
[L.F.Alday, D.Gaiotto, Y.Tachikawa]

- **Class R:**

3-manifold $M_3$ → 3d $\mathcal{N}=2$ theory

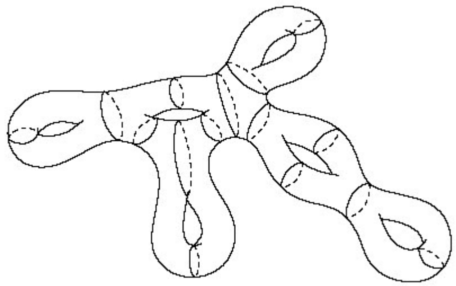
$T[M_3]$



Strings 2011

- **Class S:**

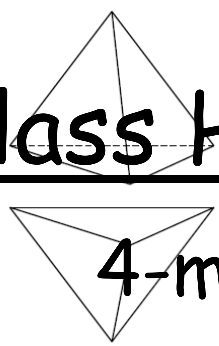
2-manifold $C$ ➡ 4d $\mathcal{N}=2$ theory **T[C]**



- **Class R:**

3-manifold $M_3$ ➡ 3d $\mathcal{N}=2$ theory **T[$M_3$]**

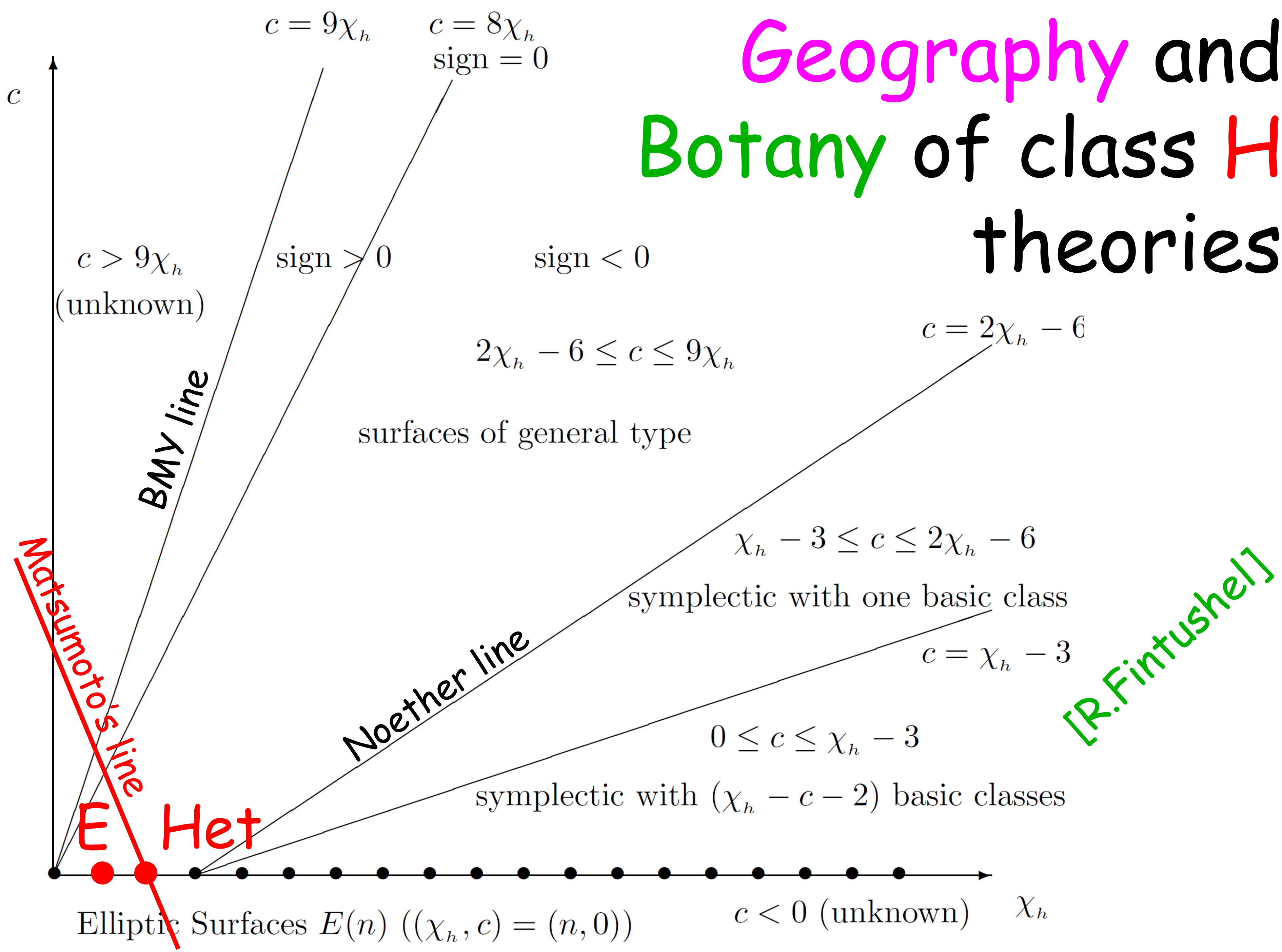
Strings 2011



- **Class H:**

4-manifold $M_4$ ➡ 2d $\mathcal{N}=(0,2)$ theory **T[$M_4$]**

# Geography and Botany of class H theories



# Motivation

- Much richer structure than (2,2) models (new branches of vacua, gauge dynamics…)

[I.Melnikov, C.Quigle, S.Sethi, M.Stern, 2012]

- (0,2) mirror symmetry

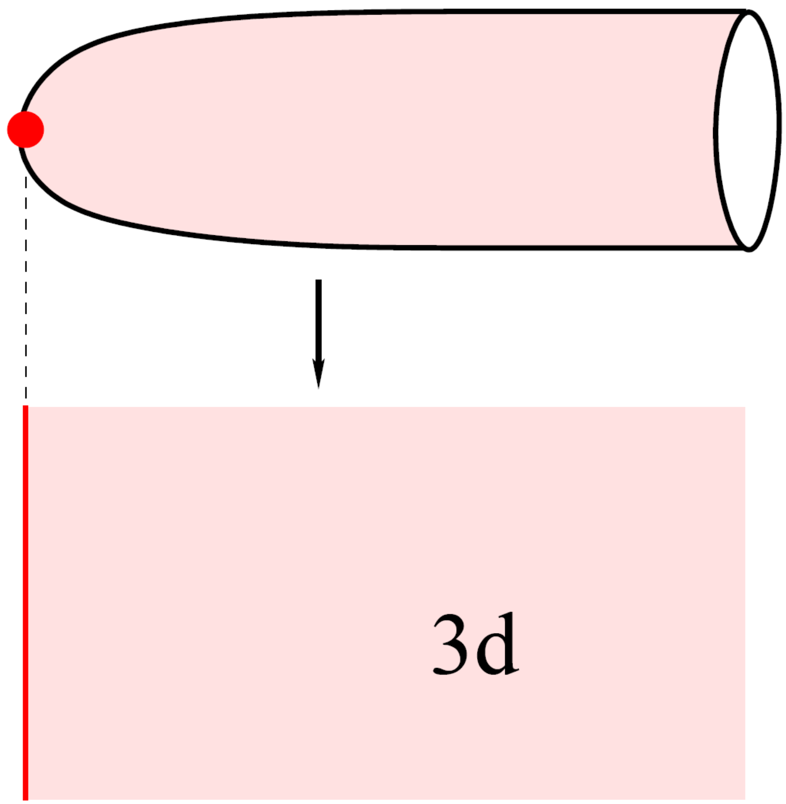
see e.g. [I.Melnikov, S.Sethi, E.Sharpe, 2012]

- Membranes (ABJM) with boundary and defect walls

- Fusion of defect ~~lines~~ walls in ~~2d~~ 3d

# Surface Operators in 4d $\mathcal{N}=1$ gauge theories

w/ D.Gaiotto and N.Seiberg




A half-BPS surface operator in 4d $\mathcal{N}=1$ gauge theory defines a half-BPS boundary condition in 3d $\mathcal{N}=2$ theory

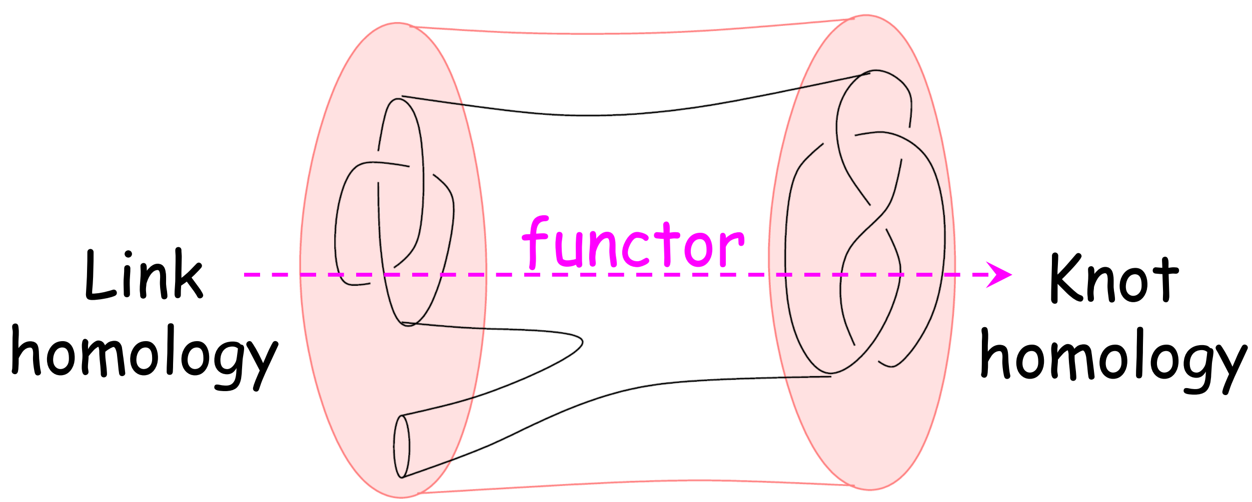
# Representations of BPS algebras

$\mathcal{H}^{(\text{closed})}_{\text{refined BPS}}$ = algebra

[J.Harvey, G.Moore]
[M.Kontsevich, Y.Soibelman]

[E.Gorsky, S.G., M.Stosic]

$\mathcal{H}^{(\text{open})}_{\text{refined BPS}}$ = module over $\mathcal{H}^{(\text{closed})}_{\text{refined BPS}}$



# Vafa-Witten partition function

$$\begin{array}{c} \text{6d } (2,0) \text{ theory} \\ \text{on } T^2 \times M_4 \end{array}$$

[C.Vafa, E.Witten]

$$\swarrow \qquad\qquad \searrow$$

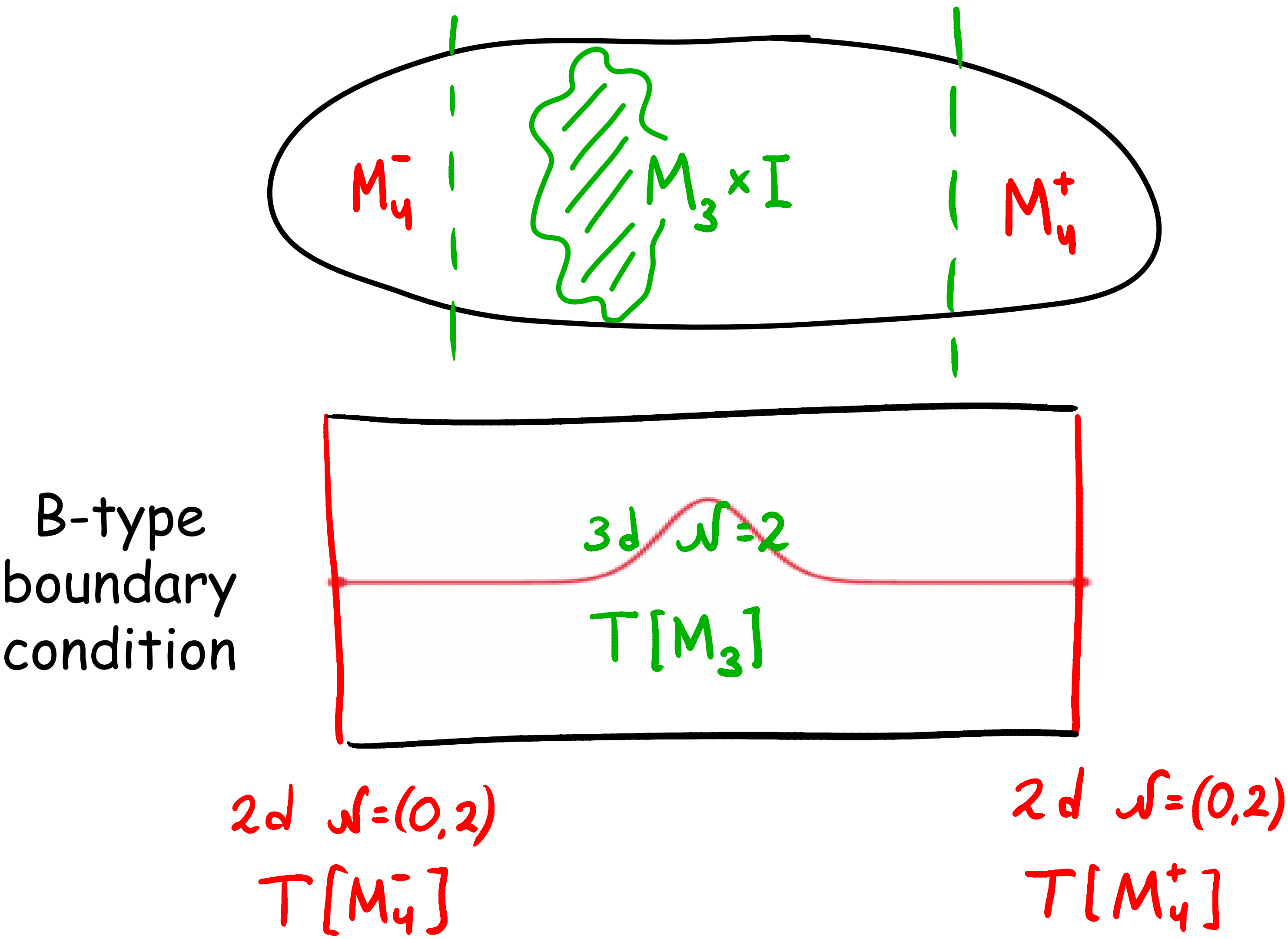
$\mathcal{N} = 4$ super-Yang-Mills on $M_4$ — 2d $(0,2)$ theory $T[M_4]$ on $T^2$

$$Z_{VW} = \sum_n x^c q^n \chi(\mathcal{M}_{n,c})$$ = "flavored" elliptic genus of the (0,2) theory

Good
~~Gluing~~ News Report #1:

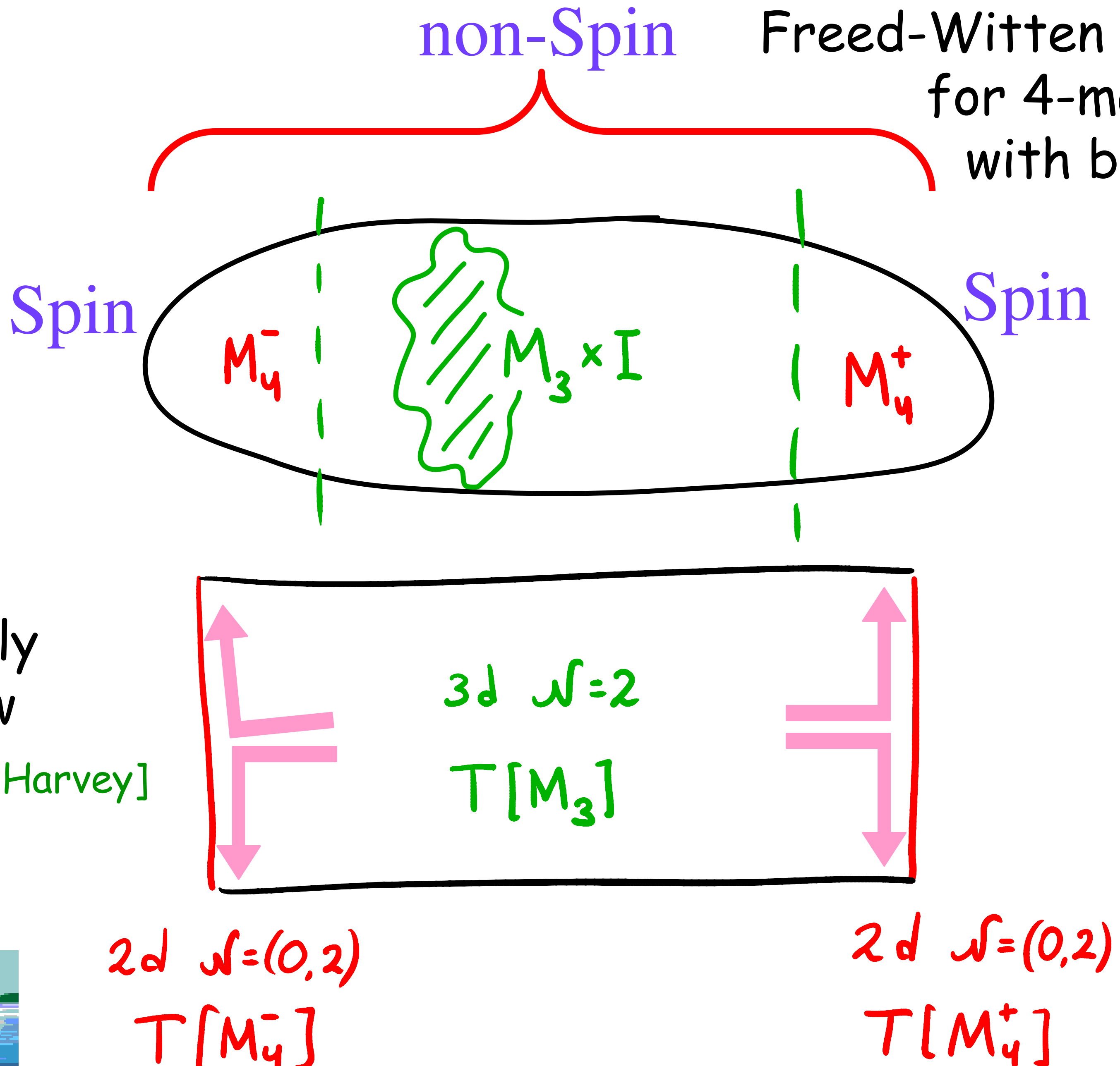
- Discrete vs continuous basis
- Integration measure = (0,2) vector multiplet superconfromal index

# Gluing



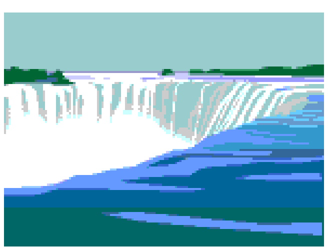
$M_4^-$
$M_3 \times I$
$M_4^+$
B-type boundary condition
3d $\mathcal{N}=2$
$T[M_3]$
2d $\mathcal{N}=(0,2)$
$T[M_4^-]$
2d $\mathcal{N}=(0,2)$
$T[M_4^+]$

# Freed-Witten anomaly for 4-manifolds with boundary

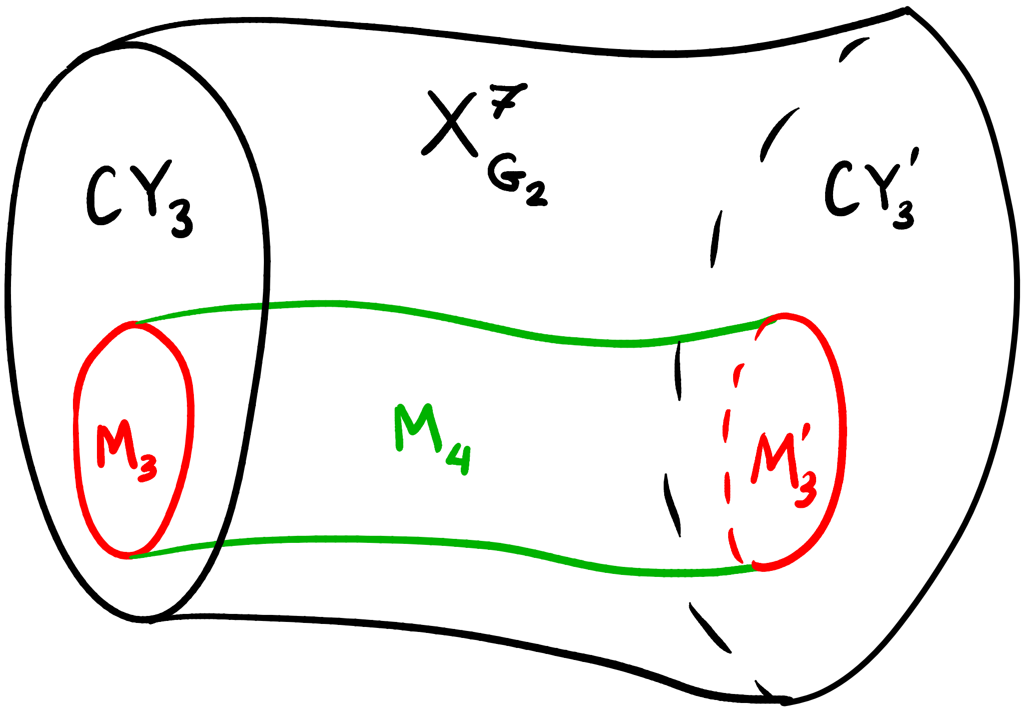



anomaly inflow

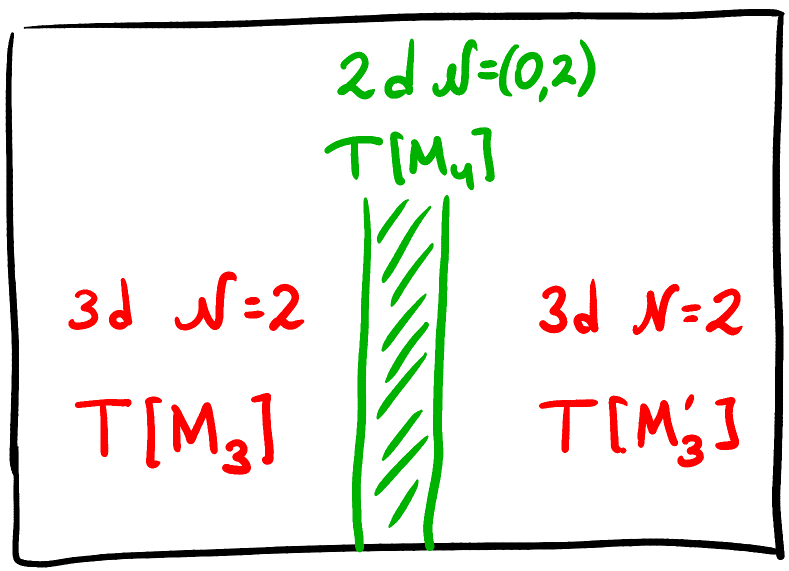
[C.Callan, J.Harvey]



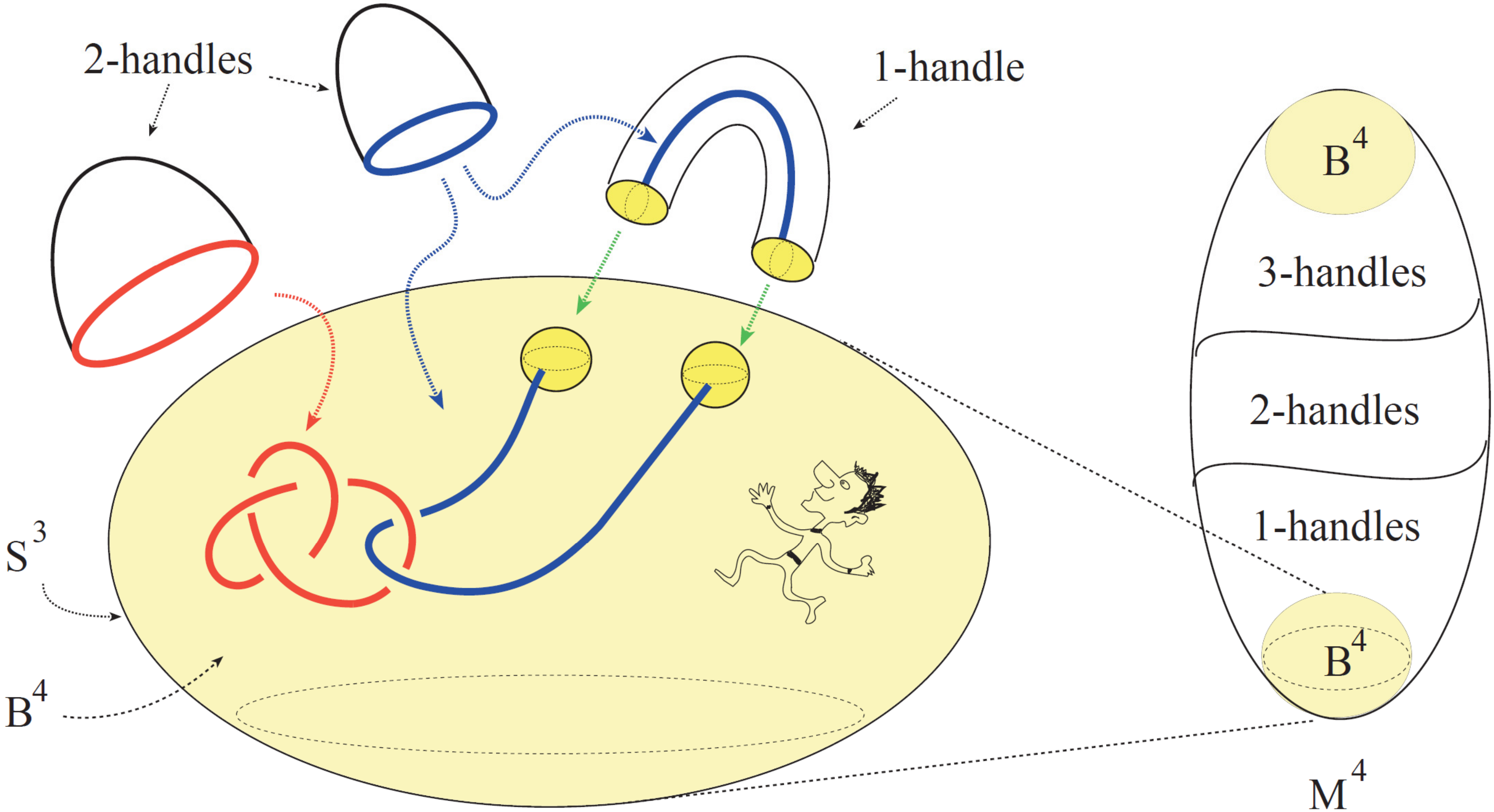
bordism




half-BPS domain wall

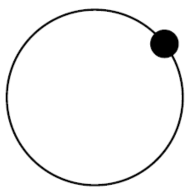


# Building blocks

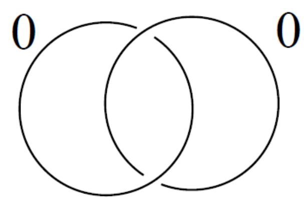



S. Akbulut, 2012

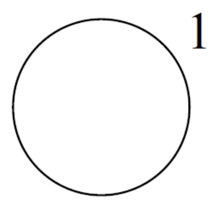
# Kirby diagrams



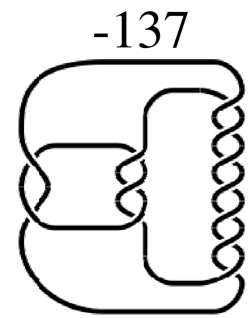
$S^1 \times S^3$




$S^2 \times S^2$




$\mathbb{CP}^2$

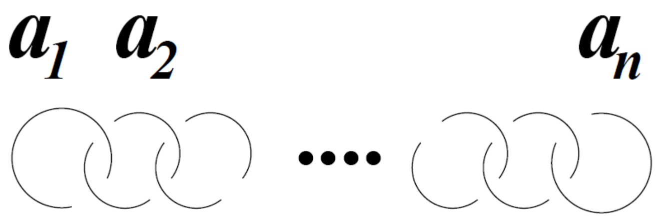



$$M_4 \quad : \qquad K_1^{a_1}\, K_2^{a_2}\, \ldots\, K_n^{a_n}$$

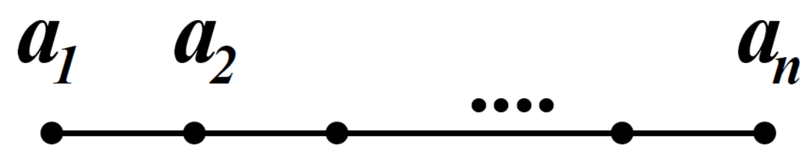
**Intersection form on $H_2(M_4;\mathbb{Z})$:**

$$Q_{ij} = \begin{cases} \mathrm{lk}(K_i, K_j), & \text{if } i \neq j \\ a_i, & \text{if } i = j \end{cases}$$

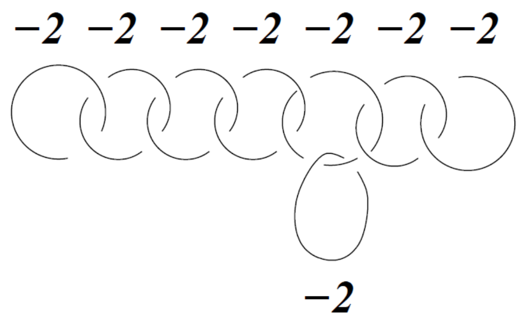
# Plumbing graphs



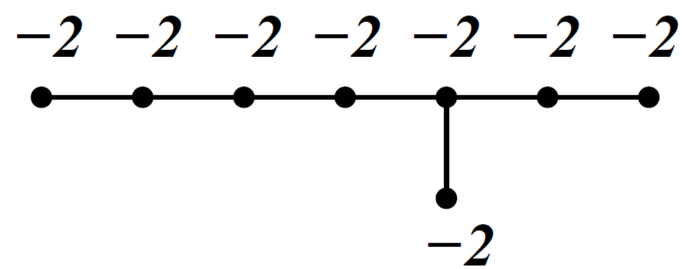

$=$



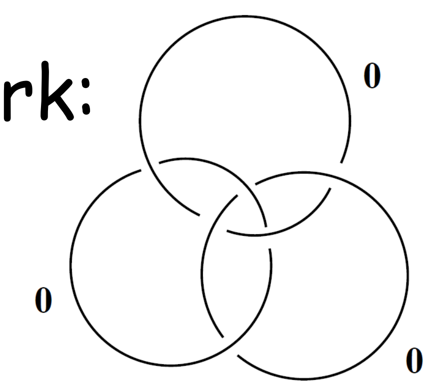

$E_8 =$



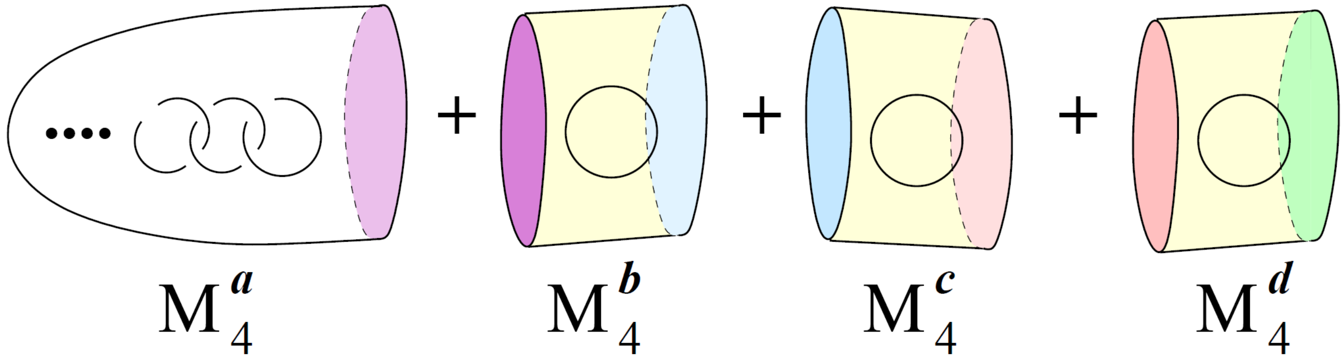

$=$



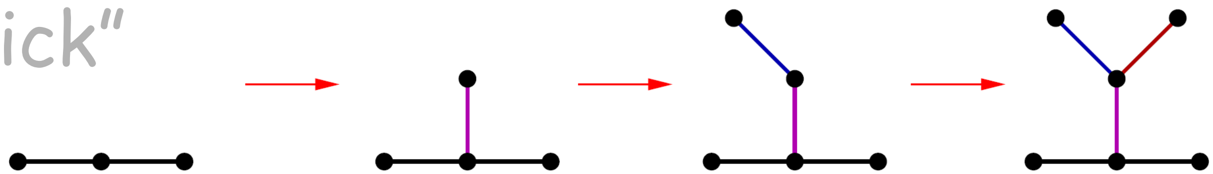

does not always work:



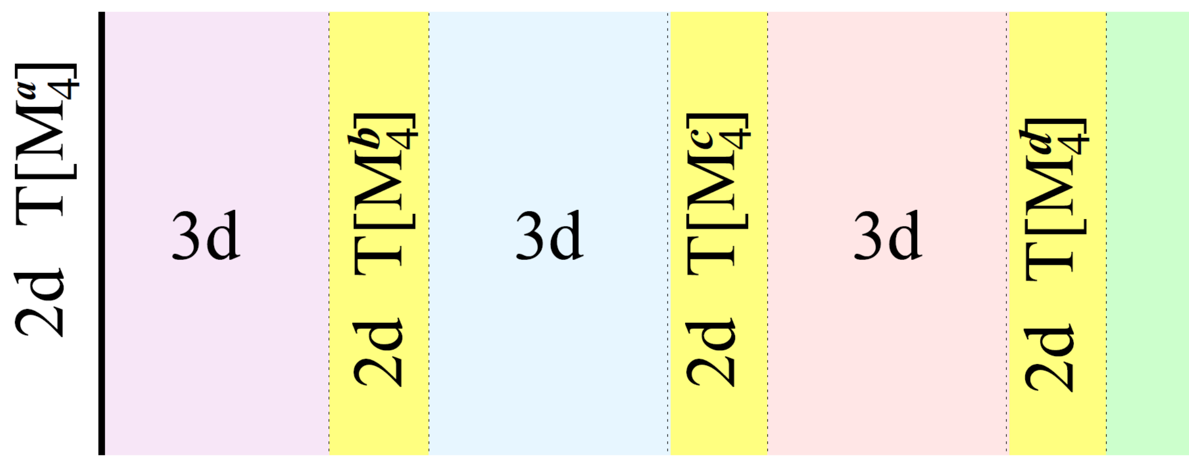

4-manifold bounded by a 3-torus




"Norman trick"



$$\chi_\rho^G = \sum_{\rho'} C_\rho^{\rho'} \chi_\rho^H$$



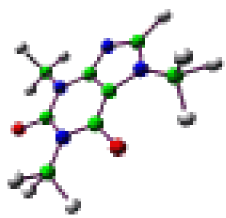

Gluing rule #2: $Z_{vw}$=coset branching function

# $\mathcal{N}=2$ quiver Chern-Simons theory

vertex $a$ ● $\longleftrightarrow$ **U(1)** Chern-Simons at level $a$

$$S = \frac{a}{4\pi}\int d^3x d^4\theta \; V\Sigma$$

$$= \frac{a}{4\pi}\int (A\wedge dA - \overline{\lambda}\lambda + 2D\sigma)$$



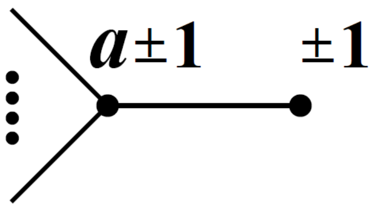
$a_i$ ●—● $a_j$ $\longleftrightarrow$

$$S = \frac{1}{2\pi}\int d^3x d^4\theta \; V_i\Sigma_j$$

edge

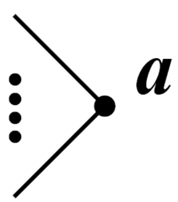
cf. [D.Belov, G.Moore]
[A.Kapustin, N.Saulina]
[J.Fuchs, C.Schweigert, A.Valentino]
:

# $\mathcal{N}=2$ quiver Chern-Simons theory

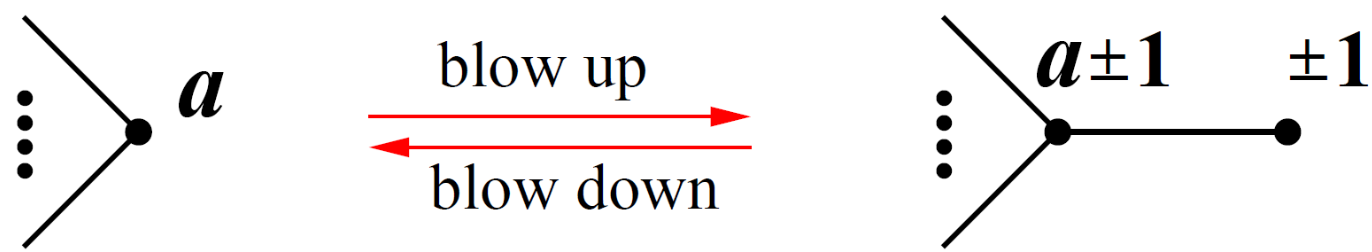



$$= \frac{1}{4\pi}\int d^4\theta \left(\pm V\Sigma + 2\widetilde{V}\Sigma + (a\pm 1)\widetilde{V}\widetilde{\Sigma} + \dots\right)$$

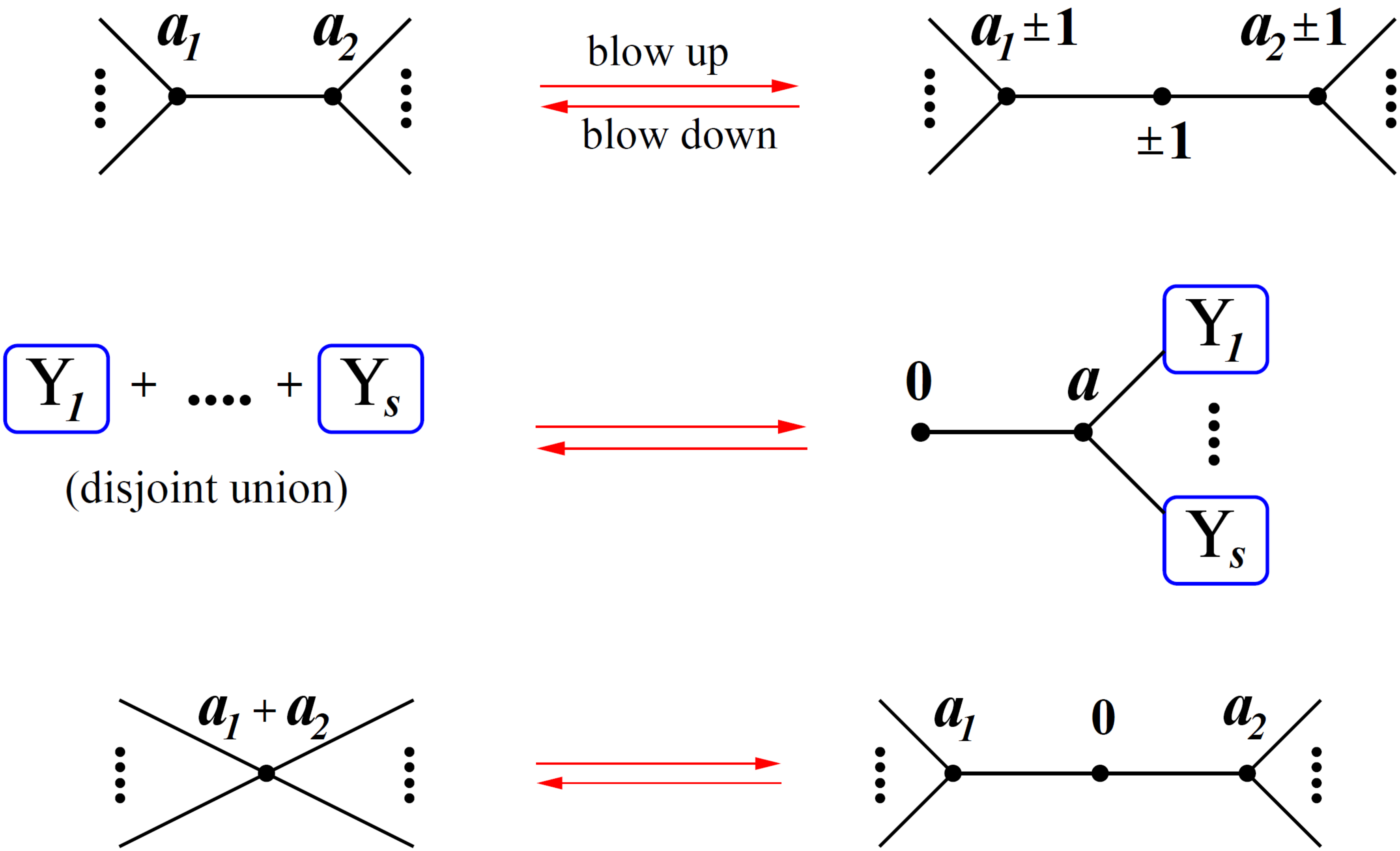
integrate out $V$ $$= \frac{1}{4\pi}\int d^4\theta \left(\pm \widetilde{V}\widetilde{\Sigma} \mp 2\widetilde{V}\widetilde{\Sigma} + (a\pm 1)\widetilde{V}\widetilde{\Sigma} + \dots\right)$$




$$= \frac{1}{4\pi}\int d^4\theta \left(a\widetilde{V}\widetilde{\Sigma} + \dots\right)$$



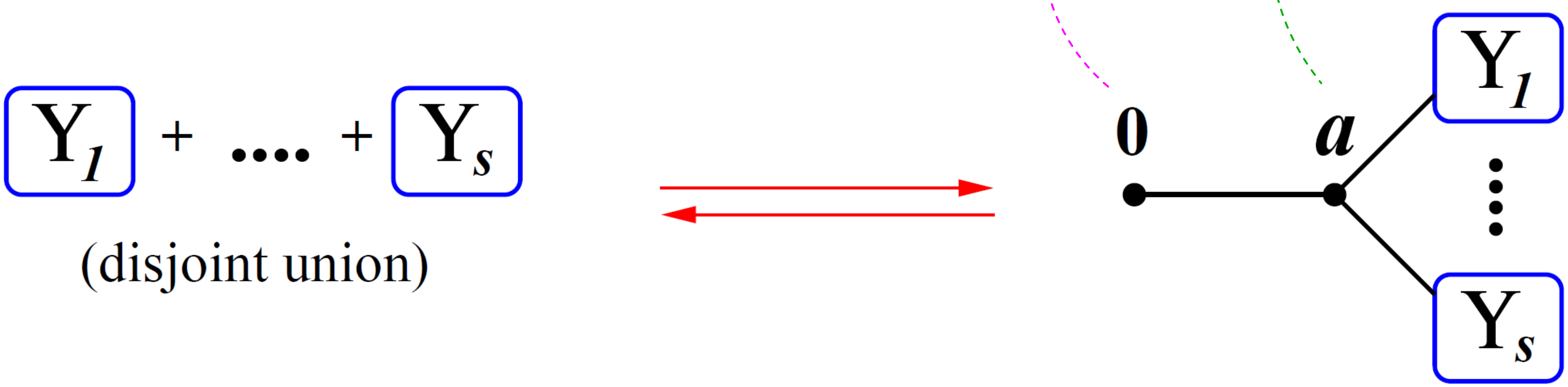
# 3d Kirby moves



# 3d Kirby moves

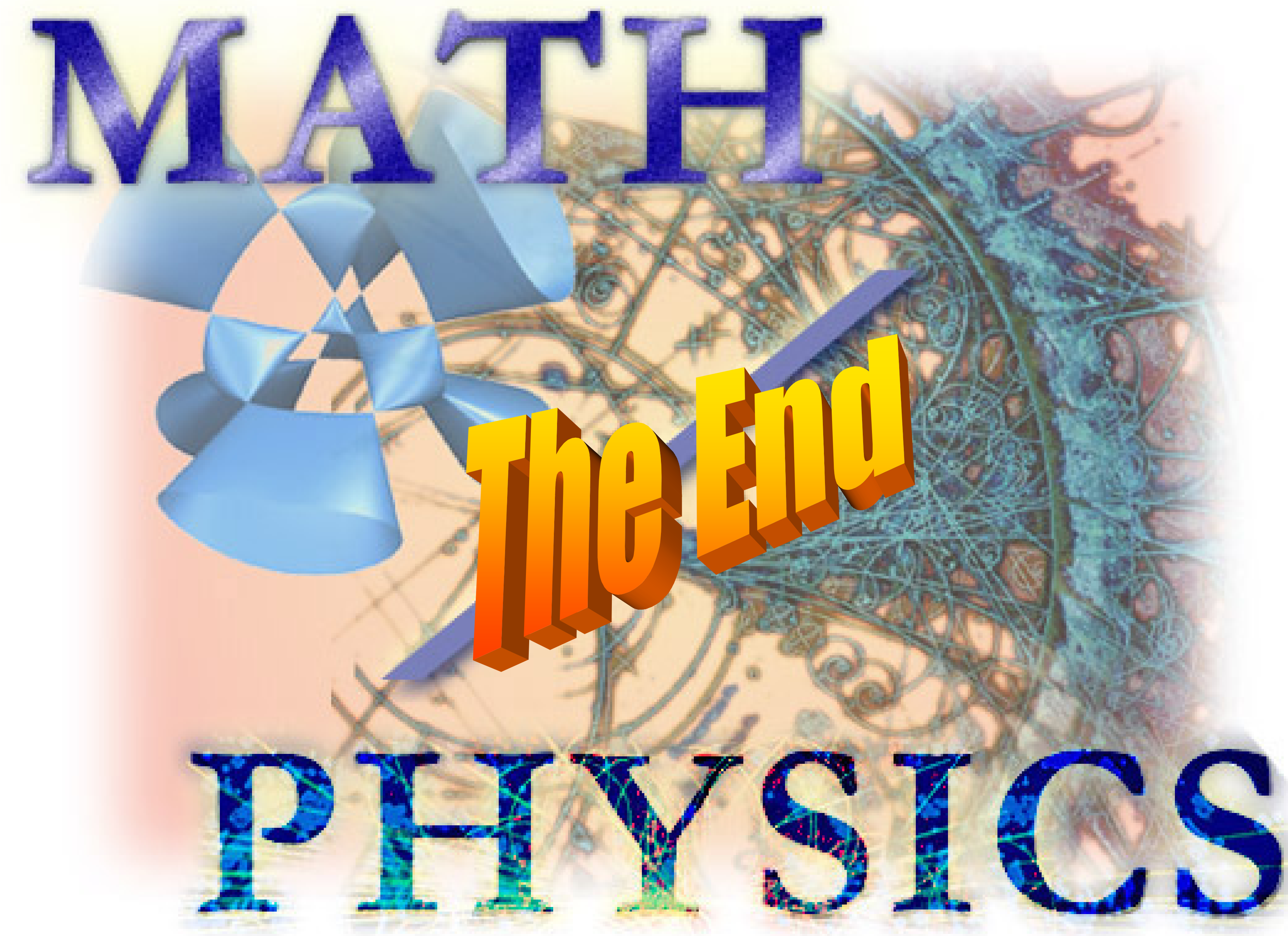
$$\mathcal{L} = \frac{1}{4\pi}\int d^4\theta \left(2V\widetilde{\Sigma} + a\widetilde{V}\widetilde{\Sigma} + \ldots\right)$$

V is Lagrange multiplier

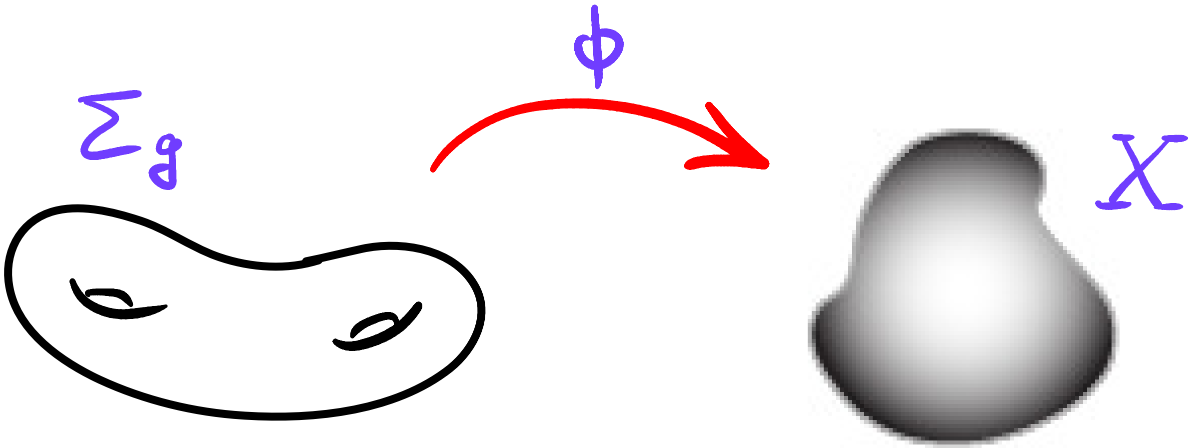



Integrating out $V$ makes $\widetilde{V}$ pure gauge
and removes all its Chern-Simons couplings

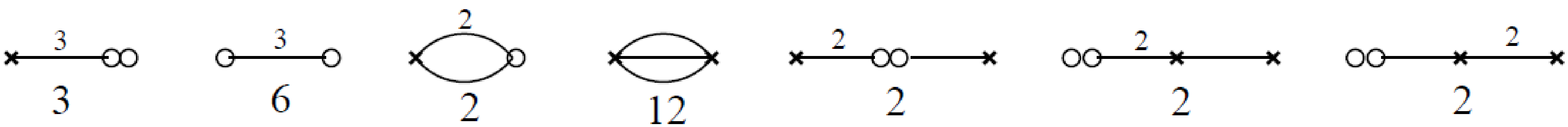
| **4-manifold** $M_4$ | **2d** $(0,2)$ **theory** $T[M_4]$ |
|---|---|
| handle slides | dualities of $T[M_4]$ |
| boundary conditions | vacua of $T[M_3]$ |
| 3d Kirby calculus | dualities of $T[M_3]$ |
| cobordism from $M_3^-$ to $M_3^+$ | domain wall (interface) between $T[M_3^-]$ and $T[M_3^+]$ |
| gluing | fusion |
| Vafa-Witten partition function | flavored (equivariant) elliptic genus |
| $Z_{VW}$(cobordism) | branching function |
| instanton number | $L_0$ |
| embedded surfaces | chiral operators |
| Donaldson polynomials | chiral ring relations |


MATH
The End
PHYSICS

# 4d Gravity = A-model




E.Witten, 1991
M.Kontsevich, 1994
N.Nekrasov, 1998
:




**ENUMERATION OF RATIONAL CURVES VIA TORUS ACTIONS**


MAXIM KONTSEVICH

Max-Planck-Institut für Mathematik, Bonn
and University of California, Berkeley


In the Woods of M-theory


Nikita Nekrasov

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*Laboratory of Physics, Harvard University, Cambridge, MA 021*

# 4D Pachner moves