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Comments on BPS States in N=4 SYM

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based on C.-M. Chang, XY [1305.6314]

The problem: count 1/16 BPS states in N=4 SYM

Earlier effort: Kinney, Maldacena, Minwalla, Raju, Berkooz, Reichmann, Simon, Kim, Lee, Grant, Grassi, ...

Why is it interesting?

There are large 1/16 BPS black holes in AdS_5.

Gutkowski, Reall, Chong, Cvetic, Lu, Pope, ...

There are large 1/16 BPS black holes in AdS_5.

And their entropies are not accounted for by the superconformal index (too small) nor by the number of BPS states in the free theory (too large).

[Kinney, Maldacena, Minwalla, Raju, ’05]

We investigate this problem in weakly coupled N=4 SYM, following [Grant, Grassi, Kim, Minwalla, '08]

1. We will identify all 1/16 BPS states in the N→∞ limit, and match precisely with BPS multi-(super) graviton states. (This was done previously within certain subsectors, but not to full generality.)

2. We conjecture that all 1/16 BPS states at finite N are of the "multi-graviton" form. Some limited evidences via computerized enumeration of cohomology.

3. If our conjecture is correct, there are not enough states to account for the Bekenstein-Hawking entropy of BPS AdS_5 black hole in supergravity - a sharp puzzle (more comments later).

Formulation of the counting problem (I)

1. We want to find all 1/16 BPS local gauge invariant operators in N=4 SYM (or equivalently, 1/16 BPS states on S^3).

2. Pick one out of the 16 supercharges, call it Q. Its Hermitian conjugate in radial quantization is a special superconformal generator S.

3. The problem of counting 1/16 BPS states, i.e. states annihilated by Q and S, is equivalent to counting Q-cohomology ($Q^2=0$).

Formulation of the counting problem (II)

4. Suffices to work with letters that are 1/16 BPS in the free theory, i.e. ones that classically saturates the BPS bound, $E=2J_L+H_1+H_2+H_3$.

5. There are 4 bosonic letters of this type,

$$\varphi^n=\Phi^{4n},\quad f=F_{++}, \qquad (n=1,2,3)$$

and 5 fermionic letters of this type,

$$\psi_n=\Psi_{n+},\quad \lambda_\alpha=\bar{\Psi}^4_\alpha, \qquad (\alpha=\pm)$$

In addition, there are 2 covariant derivatives,

$$D_\alpha\equiv D_{+\alpha}\ .$$

Formulation of the counting problem (III)

6. And constraints (equation of motion)

$$[D_\alpha, D_\beta] = \varepsilon_{\alpha\beta} f\,, \qquad D_\alpha \lambda^\alpha = [\varphi^n, \psi_n]\ .$$

7. Introduce commuting generating parameter z^α. Write

$$\varphi^n(z) = \sum (z^\alpha D_\alpha)^k \varphi^n / k!\ ,\ \text{and similarly } \psi_n(z),\ f(z).$$

Also, $\lambda(z) = \sum (z^\alpha D_\alpha)^k z^\beta \lambda_\beta / (k+1)!$

8. Introduce anti-commuting generating parameter θ_n. Write

$$\Psi(z,\theta) = \lambda(z) + \theta_n \varphi^n(z) + \varepsilon^{nmp} \theta_n \theta_m \psi_p(z) + \theta_1 \theta_2 \theta_3 f(z)\ .$$

Formulation of the counting problem (IV)

9. Write $\mathcal{Z}\equiv(z,\theta)$. Q-action now takes the concise form

$$\{Q, \Psi(\mathcal{Z})\} = \Psi(\mathcal{Z})^2.$$

The only constraint on $\Psi(\mathcal{Z})$ is

$$\Psi(0) = 0.$$

10. All gauge invariant operators of zero twist are made out of products of derivatives of $\Psi(\mathcal{Z})$, then restricted to $\mathcal{Z}=0$.

Q-cohomology at N=∞

They are products of single trace operators of the following form:

$$\prod_{\alpha}\partial_{z_\alpha}^{p_\alpha}\prod_{n}\partial_{\theta_n}^{q_n}\mathrm{Tr}\Big[\prod_{\beta}(\partial_{z_\beta}\Psi)^{k_\beta}\prod_{k}(\partial_{\theta_k}\Psi)^{m_k}\Big]\Big|_{z=\theta=0}$$

These precisely agree with the counting of free 1/16 BPS multi-gravitons in AdS_5xS^5.

Finite N

There are trace relations among the operators that describe multi-graviton states in the infinite N limit. We will still refer to such operators as "multi-graviton" operators at finite N.

Are there **new** Q-cohomology classes due to trace relations?

Let's try...

For SU(N) theory, consider a fermionic letter X (X could be $\partial_{z^+}\Psi$, for instance). There is a trace relation of the form

$$\mathrm{Tr}(X^{2N+1}) = \text{multi-trace}.$$

Now since $Q\mathrm{Tr}(\partial^2_{z^+}\Psi(\partial_{z^+}\Psi)^{2N-1}) = \mathrm{Tr}((\partial_{z^+}\Psi)^{2N+1})$, we can try to construct a new Q-cohomology class using the operator $\mathrm{Tr}(\partial^2_{z^+}\Psi(\partial_{z^+}\Psi)^{2N-1})$. But, there is also a trace relation of the form

$$\mathrm{Tr}(X^{2N-1}Y) = \text{multi-trace}.$$

In the end, no new Q-cohomology arises this way.

Computer tests

Next, we try the most straightforward search of new cohomology classes, by enumerating Q-closed and Q-exact operators at low levels for SU(2), SU(3), SU(4) theories, and ask if all Q-cohomology classes are accounted for by the "multi-graviton" type operators.

This test is difficult to carry out to high levels, due to the computational complexity.

We did not find any new Q-cohomology class, in all cases tested.

Some SU(2) and SU(3) examples

Here we label the operator by the charge vector that counts the number of ∂_z's and ∂_θ's. The Q-cohomology in each charge sector is then graded by the number of Ψ's, which ranges from 2 to the total number of ∂'s.

charges	N = 2	N = 3	N = ∞
[4,4; 0,0,0]	(1, 0, 4, 0, 0, 0, 0)	(1, 0, 5, 0, 1, 0, 0)	(1, 0, 5, 0, 2, 0, 1)
[5,4; 0,0,0]	(1, 0, 5, 0, 0, 0, 0, 0)	(1, 0, 6, 0, 3, 0, 0, 0)	(1, 0, 6, 0, 4, 0, 1, 0)
[5,0; 4,0,0]	(0, 0, 3, 0, 4, 0, 0, 0)	(0, 0, 3, 10, 6, 2, 0, 0)	(0, 0, 4, 11, 10, 2, 0, 0)
[4,0; 5,0,0]	(0, 0, 0, 0, 6, 0, 0, 0)	(0, 0, 0, 5, 10, 8, 1, 0)	(0, 0, 0, 6, 19, 14, 2, 0)
[0,0; 5,4,0]	(0, 0, 0, 0, 0, 0, 1, 0)	(0, 0, 0, 0, 0, 2, 7, 5)	(0, 0, 0, 0, 0, 11, 44, 40)
[0,0; 2,2,2]	(0, 0, 4, 0, 1)	(0, 0, 7, 11, 5)	(0, 0, 8, 19, 16)
[0,0; 3,3,1]	(0, 0, 1, 0, 3, 0)	(0, 0, 2, 9, 13, 5)	(0, 0, 2, 14, 35, 23)
[2,2; 1,1,0]	(3, 0, 13, 0, 0)	(3, 3, 22, 7, 1)	(3, 3, 23, 13, 5)
[1,1; 2,2,0]	(0, 0, 12, 0, 1)	(0, 1, 19, 20, 6)	(0, 1, 22, 33, 20)
[3,2; 1,1,0]	(3, 0, 26, 0, 0, 0)	(3, 3, 35, 17, 7, 0)	(3, 3, 36, 23, 16, 3)
[1,1; 3,2,0]	(0, 0, 9, 0, 6, 0)	(0, 0, 12, 33, 30, 8)	(0, 0, 13, 46, 78, 40)
[1,1; 1,1,1]	(4, 0, 10, 0)	(4, 6, 20, 5)	(4, 6, 24, 11)
[2,1; 1,1,1]	(4, 0, 28, 0, 0)	(4, 6, 44, 21, 3)	(4, 6, 48, 35, 13)
[1,1; 2,1,1]	(1, 0, 27, 0, 1)	(1, 4, 41, 34, 8)	(1, 4, 47, 55, 28)
[1,1; 3,1,1]	(0, 0, 26, 0, 10, 0)	(0, 1, 32, 71, 52, 11)	(0, 1, 36, 95, 131, 58)

Some more SU(2) examples

Here we label the operator by the charge vector that counts the number of ∂_z's and ∂_θ's. The Q-cohomology in each charge sector is then graded by the number of Ψ's, which ranges from 2 to the total number of ∂'s.

charges	N = 2	N = ∞
[6,5; 0,0,0]	(1, 0, 9, 0, 1, 0, 0, 0, 0, 0)	(1, 0, 10, 0, 12, 0, 4, 0, 1, 0)
[3,3; 3,0,0]	(0, 0, 23, 0, 19, 0, 0, 0)	(0, 1, 26, 53, 90, 65, 28, 8)
[3,0; 3,3,0]	(0, 0, 3, 0, 21, 0, 1, 0)	(0, 0, 4, 30, 121, 158, 83, 11)
[0,0; 3,3,3]	(0, 0, 0, 0, 4, 0, 3, 0)	(0, 0, 0, 3, 49, 175, 258, 131)
[1,1; 2,2,1]	(0, 0, 41, 0, 10, 0)	(0, 1, 58, 128, 170, 72)
[2,1; 2,1,1]	(1, 0, 58, 0, 5, 0)	(1, 4, 81, 118, 114, 35)
[2,2; 1,1,1]	(4, 0, 67, 0, 1, 0)	(4, 6, 94, 92, 77, 20)
[2,2; 2,1,1]	(1, 0, 114, 0, 28, 0, 0)	(1, 4, 140, 242, 382, 237, 60)
[2,1; 2,2,1]	(0, 0, 77, 0, 42, 0, 0)	(0, 1, 95, 236, 465, 352, 100)
[1,1; 2,2,2]	(0, 0, 46, 0, 43, 0, 1)	(0, 0, 54, 191, 508, 515, 199)

In each case, no new cohomology class found.

We are led to **conjecture** that there are **no new** Q-cohomology classes, i.e. all 1/16 BPS operators are of the multi-graviton type, namely products of

$$\prod_\alpha \partial_{z_\alpha}^{p_\alpha} \prod_n \partial_{\theta_n}^{q_n} \mathrm{Tr}[\prod_\beta (\partial_{z_\beta}\Psi)^{k_\beta} \prod_k (\partial_{\theta_k}\Psi)^{m_k}]\Big|_{z=\theta=0}$$

In particular, this would imply that the number of 1/16 BPS operators of a given dimension in the SU(N) theory cannot be more than the number of such operators in the $N\to\infty$ limit (free gravitons).

The entropy of the latter grows like $S\sim E^{5/6}$. One can also show that the entropy of "multi-graviton" operators at finite N is bounded by $S\lesssim N^{1/3}E^{2/3}$. Not enough states to account for large black hole entropy $S\sim N^2$ when $E\sim N^2$.

Some formality

In mathematical terms, our Q-cohomology classes are given by the relative Lie algebra cohomology (with coefficients in $\mathbb{C}$)

$$H^*(\mathfrak{g}_N, sl_N)$$

where $\mathfrak{g}_N$ is the (infinite dimensional) Lie algebra of $\mathbb{C}[z_+,z_-]\otimes\Lambda[\theta_1,\theta_2,\theta_3]\otimes sl_N$, with the subalgebra $sl_N\subset\mathfrak{g}_N$. The grading of the cohomology is the degree in Ψ.

The inclusion $\mathfrak{i}: \mathfrak{g}_{N-1} \to \mathfrak{g}_N$ induces a map on the cohomology

$$\mathfrak{i}^*: H^*(\mathfrak{g}_N, \mathfrak{sl}_N) \to H^*(\mathfrak{g}_{N-1}, \mathfrak{sl}_{N-1})$$

Our conjecture amounts to the statement that $\mathfrak{i}^*$ is surjective, for all N.

Puzzle

We are not producing the expected Bekenstein-Hawking entropy of large 1/16 BPS black hole from the counting of 1/16 BPS operators in N=4 SYM at weak coupling.

Possibility 1

The conjecture is false. Perhaps we did not test to high enough level in the SU(2) and SU(3) examples.

Should be possibile to prove or disprove the conjecture by analyzing (a generalization of) the Hochschild-Serre spectral sequence associated with the embedding $\mathfrak{g}_{N-1} \rightarrow \mathfrak{g}_N$.

Possibility 2

The number of 1/16 BPS states jump as the coupling increases. We see no evidence for this, however.

Possibility 3

1/16 BPS black hole solutions in supergravity are destroyed by stringy corrections, and such black holes do not exist in the full string theory on $AdS_5 x S^5$.

In any case, the existence of the 1/16 BPS black hole solutions in supergravity implies at least large number of near-1/16 BPS states at strong coupling, at dimension ~ N^2. [Berkooz, Reichmann '08]

This is a fascinating regime of N=4 SYM to explore.